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SPECIAL
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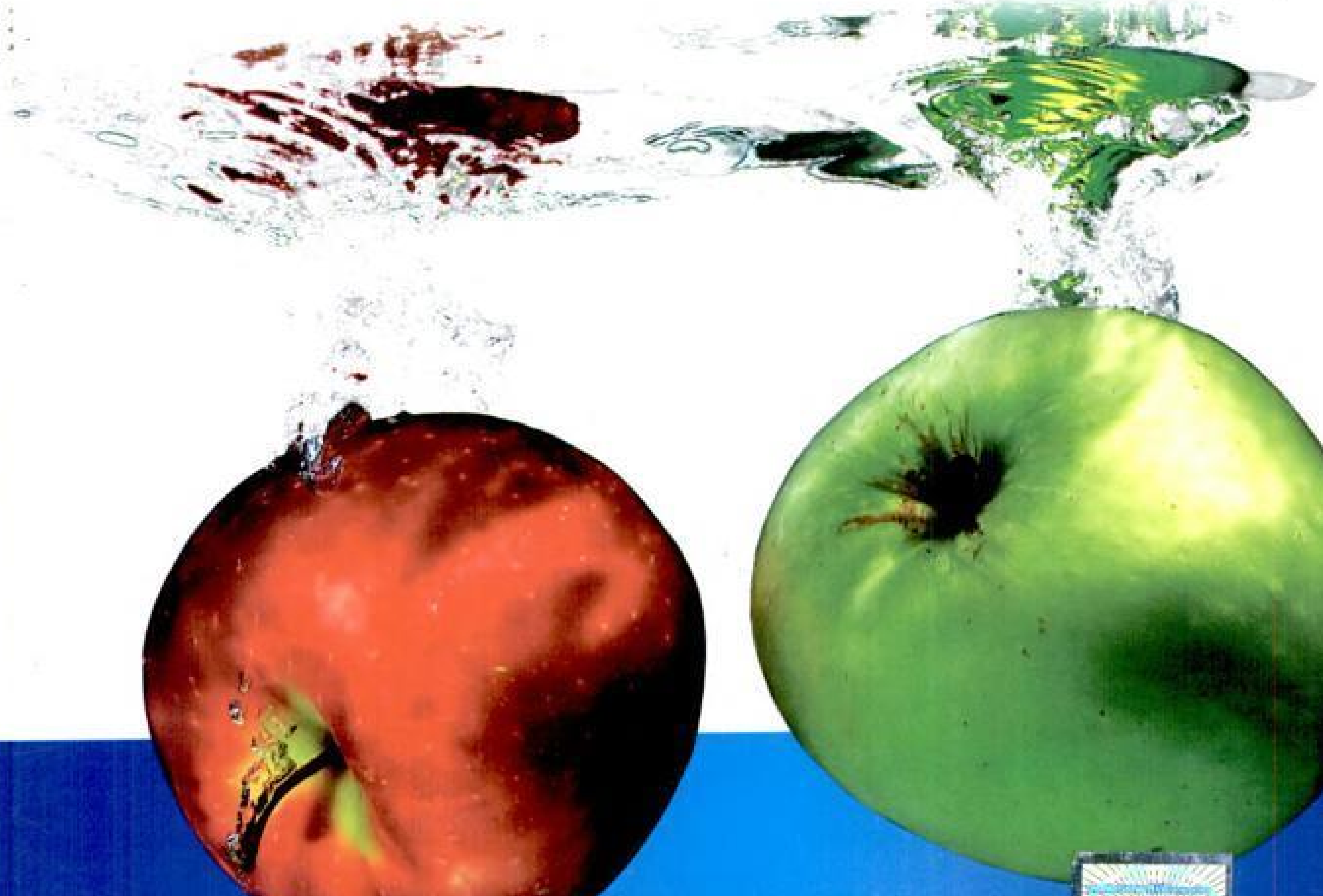
EDITION

ENGINEERING MECHANICS

Revised Fourth Edition



S TIMOSHENKO | D H YOUNG | J V RAO

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About the Adaptation Author

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Preface to the Adapted Edition

This book is the product of more than half a century of innovation in Engineering Mechanics education. When the first edition of Engineering Mechanics by S. Timoshenko and D.H. Young appeared in 1937, it was revolutionary among engineering mechanics textbooks with its emphasis on the fundamental principles of mechanics and how to apply them. The success of Engineering Mechanics with generations of students and educators throughout the world is a testament to the merits of this approach. In this revised fourth edition, SI units, which are most frequently used in mechanics, are introduced in Chapter 1 and are used throughout.

Objective

The main objective of a first course in mechanics should be to build a strong foundation, to acquaint the student with as many general methods of attack as possible, and to illustrate the application of these methods to practical engineering problems. However, it should avoid routine drill in the manipulation of standardized methods of solution. Such are the aims of this book. This text is designed for the first course in statics and dynamics offered in sophomore or junior year. It is hoped that this text will help the instructor achieve this goal.

General Approach

Scalar approach is used throughout the presentation of statics and dynamics. In Part One, statics, chapters are based on Force systems except principle of virtual work. In dynamics, emphasis is on rectilinear translation, curvilinear translation, rotation of a rigid body about a fixed axis and plane motion. Finally relative motion is dealt with in Part Two.

New to This Edition

While retaining the well-received approach and organization of the previous edition, the revised fourth edition offers the following new features and improvements:

- Each topic ends with a summary of the material covered in it.

- A set of review questions is included at the end of each topic. Short questions are added for the benefit of students which are useful for university examination.
- Multiple choice questions are given at the end of each topic to help the student prepare for competitive exams like IES, IAS (Prelims), GATE, etc.
- Important terms and concepts are added in each topic.
- Important formulae are added in each topic.
- Numerous new examples and new problems are added.

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I want to extend my heartful thanks to my colleagues at Bapatla Engineering College, especially Asst. Prof. J. Girish, Prof. A.V. Narayanappa, for many stimulating discussions about engineering mechanics pedagogy and for their support and encouragement during the adaptation of this book. I am equally indebted to the Bapatla Engineering College students who have helped me learn what good teaching and good writing are, by showing me what works and what doesn't. I want to express special thanks to Sri Muppalaneni Sheshagiri Rao, President, Bapatla Educational Society and Prof. G. N. Rao, Principal, BEC, Prof. K. L. Prasad, H.O.D. Civil, BEC for their support while writing this book. I also want to express special thanks to Vibha Mahajan, Shukti Mukherjee and Mini Narayanan of Tata McGraw-Hill for their superb editorial guidance and vision. I want to thank my parents, wife and children for their support.

Feedback from professors and students, especially concerning errors or deficiencies in this edition are welcome. Comments and suggestions for further improvement of the text will be greatly appreciated.

J V Rao

Preface

The importance of mechanics in the preparation of young engineers for work in specialized fields cannot be overemphasized. The demand from industry is more and more for young men who are soundly grounded in their fundamental subjects rather than for those with specialized training. There is good reason for this trend: The industrial engineer is continually being confronted by new problems, which do not always yield to routine methods of solution. The man who can successfully cope with such problems must have a sound understanding of the fundamental principles that apply and be familiar with various general methods of attack rather than proficient in the use of anyone. It seems evident, then, that university training in such a fundamental subject as mechanics must seek to build a strong foundation, to acquaint the student with as many general methods of attack as possible, to illustrate the application of these methods to practical engineering problems, but to avoid routine drill in the manipulation of standardized methods of solution. Such are the aims of this book.

The content of the book is somewhat wider than can be covered in two courses of three semester hours or five quarter hours each. At the end of the discussion of statics, for example, there is a chapter on the principle of virtual work. The use of this principle results in great simplification in the solution of certain problems of statics, and it seems desirable to acquaint the student with its possibilities. At the end of the discussion of dynamics, there is a short chapter on relative motion, together with applications to engineering problems. These chapters can easily be omitted without introducing any discontinuity if there is insufficient time for them. Where time will not permit their consideration, they at least serve the purpose of indicating to the student that he has not exhausted the possibilities of the subject in his first encounter with it. Also, it is hoped that such material will be of value to those students especially interested in mechanics.

In many of our engineering schools, statics is given during the second semester of the sophomore year, before the student has studied integral calculus. For this reason Part One of this volume has been so written that, except for one or two sections that can easily be omitted, no knowledge of mathematics beyond the differential calculus is required. However, a free use of mathematics is made within these limits. Statics is probably the first course wherein the student has a chance to make practical use of his training in mathematics, and it seems important that he be not only given the opportunity but encouraged to use it to the full extent of its applicability.

The situation is usually quite different with dynamics. In some schools, for instance, this course does not immediately follow statics but is taken after strength of materials. Thus the students are more mature, and it seems justifiable in Part Two to make free use of the calculus and even some use of elementary differential equations. In this latter respect, however, the solutions are discussed in sufficient detail so that the student without special preparation in differential equations need have no difficulty.

Throughout Part Two the equations of motion are presented and handled as differential equations. Dynamics is not a subject to be handled superficially, and a too-arduous attempt to simplify its presentation can easily result in the fostering of false notions in the mind of the beginner. Besides helping to forestall such possible misconceptions, the use of the differential equation of motion, as such, possesses several other advantages: (1) It makes it possible, at the outset, to place proper emphasis upon the inherent difference between dynamical problems involving known motion and those involving known acting forces. (2) It makes practicable the discussion of certain problems of dynamics (such as vibration problems) which otherwise could be handled only in a very cumbersome manner, if at all. (3) It gives the student a foundation in dynamics upon which he can successfully build if he desires to pursue advanced study or to read current literature on the subject.

Since the student usually has his greatest difficulty in applying the principles and theorems that he has just learned to specific situations, special attention has been given to the selection and treatment of a series of illustrative examples at the end of each article. The purpose of these examples is twofold: (1) They are sometimes used as a medium of presentation of material not included in the text proper. (2) They are designed to set an example to the student in logical methods of approach to the solution of engineering problems. It is hoped that the examples will help the student to bridge the gap between mere cognizance of the general principles and the ability to apply them to concrete problems. Mastery in this respect is the. True goal of engineering education. The examples warrant as much attention from the student as the text material proper.

The solution of a problem in mechanics usually consists of three steps: (1) the reduction of a complex physical problem to such a state of idealization that it can be expressed algebraically or geometrically; (2) the solution of this purely mathematical problem; and (3) the interpretation of the results of the solution in terms of the given physical problem. It is too often the case that the student's attention is called only to the second step so that he does not see clearly the connection between this and the true physical problem. By successive development of these three steps in the solution of each illustrative example, it is hoped to lead the student to a realization of the full significance of mechanics, and also to encourage him to approach the solution of his own problems in a similar way.

Many of the illustrative examples are worked out in algebraic form, the answers being given simply as formulas. When numerical data are given, their substitution is made only in the final answer at the end. Such a procedure possesses several advantages, one of which is the training the student gets in reliable methods of checking answers. Two of the most valuable aids in checking the solution of a problem are the "dimensional check" and the consideration of certain limiting cases as logical extremes. The opportunity of making either of these checks is lost when given numerical data are substituted at the beginning of the solution. Another advantage of the algebraic solution is that it greatly enriches the possibilities of the third step in the solution of the problem, namely, significance of results. Finally, the algebraic solution is preferable if proper attention is to be given to numerical calculations, for only by having the result in algebraic form can it be seen with what number of figures any intermediate calculation must be made in order to obtain a desired degree of accuracy in the final result.

Since the first edition of "Engineering Mechanics" appeared in 1937, the authors' "Theory of Structures" and "Advanced Dynamics" have been published, and these later volumes now contain some of the more advanced material that was originally in "Engineering Mechanics." It is hoped that the three volumes taken together represent a fairly complete treatment of engineering mechanics and its applications to problems of modem structures and machines, at the same time leaving the present volume better suited to the undergraduate courses in statics and dynamics as given in our engineering schools today.

In the preparation of this fourth edition, the entire book has been thoroughly revised. In doing this, the authors have had these objectives: (1) simplification of the text proper, (2) improved arrangement of subject matter, and (3) deemphasis of the algebraic treatment of problems. Almost all problems throughout the book are now given with numerical data and numerical answers. Furthermore, the problem sets have been completely revised, and they contain a high percentage of new problems. The problems preceded by an asterisk present special difficulties of solution.

Various textbooks have been used in the preparation of this book, particularly in the selection of problems. In this respect, special acknowledgment is due the book "Collection of Problems of Mechanics," edited by J. V. Mestscherski (St. Petersburg, 1913), in the preparation of which the senior author took part. The authors also take this opportunity to thank their colleagues at Stanford University for many helpful suggestions in regard to this revision, in particular, Prof. Karl Klotter, who read some portions of the revision and made many valuable suggestions for improvement in this edition.

S TIMOSHENKO
D H YOUNG

Part One

STATICS

1

Introduction

1.1 ENGINEERING MECHANICS

The importance of mechanics in the preparation of young engineers for work in specialized fields cannot be overemphasized. Therefore, it is appropriate to begin with a brief explanation on the meaning of the term engineering mechanics and the role of this course in engineering education. Before defining engineering mechanics, we must consider the similarities and differences between science and engineering. In general terms, science is the knowledge that comes from observing facts about the universe carefully, carrying out experiments and making statements that are always true in particular conditions. On the other hand, engineering is the application of mathematics and science to the design and manufacture of items that benefit humanity. Design is the basic concept that distinguishes engineers from scientists. Accreditation Board of Engineering and Technology (ABET) defines engineering design as the process of devising a system, component or process to meet desired needs.

Mechanics may be defined as the science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Engineering mechanics is the branch of engineering that applies the principles of mechanics to design, which must take into account the effect of forces. The goals of engineering mechanics courses is to build a strong foundation, to acquaint the student with as many general methods of attack as possible, to illustrate the application of these methods to practical engineering problems, but to avoid routine drill in the manipulation of standardized methods of solution.

Depending upon the nature of the problems treated, mechanics is divided into statics and dynamics. Statics is the study of forces and conditions of equilibrium of material bodies subjected to the action of forces. Dynamics is the study of motion of rigid bodies and their correlation with the forces causing them. Dynamics is divided into kinematics and kinetics. Kinematics deals with the space–time relationship of a given motion of a body and not at all with the forces that cause the motion. Kinetics studies the laws of motion of material bodies under the action of forces or Kinetics is the study of the relationship between the forces and the resulting motion.

Of course, engineering mechanics is an integral component of the education of engineers whose disciplines are related to the mechanical behaviour of bodies or fluids. Such behaviour is of interest to aeronautical, civil, chemical, electrical, mechanical, metallurgical, mining and textile engineers. A sound training in engineering mechanics continues to be one of the most important aspects of engineering education due to the interdisciplinary character of many engineering applications (e.g., spaceships, robotics and manufacturing). It is appropriate to conclude that the subject of engineering mechanics lies at the core of all engineering analysis.

Idealisation in Mechanics

Mathematical models or idealisations are used in mechanics to simplify the application of theory. Now, we will define some of the important idealisations. Others will be discussed at points where they are needed.

Continuum Continuum may be defined as a continuous distribution of matter with no voids or empty spaces. Each body is made up of atoms and molecules. The matter is assumed as continuously distributed since the behaviour of atoms and molecules are too complex to deal with. It is used to study the measurable behaviour.

Particle In the abstract sense, a particle is a point mass or a material point. A body whose dimensions can be neglected in studying its motion or equilibrium may be treated as a particle.

Examples While studying the planetary motion, sun and planets; moon as seen from the earth; a satellite orbiting the earth and seen by an observer on the earth are considered as particles.

System of Particles A system of particles is an idealization of point masses. A system of particles is constituted, when two or more bodies are represented by particles and are dealt with together.

Examples: Planetary system; the structure of atom, i.e., electron–proton–neutron.

Rigid Body A rigid body is the one in which the distance between any two arbitrary points is invariant. Actually, solid bodies are not rigid, but deform under the action of forces. It is assumed to be rigid, if the deformation is negligible compared to the size of the body.

Examples: A wheel of a car; a lever supporting two weights at its ends.

Basic Concepts

For the investigation of problems of engineering mechanics we must introduce the concept of space, mass, time and force. The basic concepts used in mechanics are space, time, mass and force. These basic concepts cannot be truly defined but are developed for axiomatic thinking and mutual understanding.

Space Space refers to the geometric region occupied by bodies. The positions of bodies are described by linear and angular measurements relative to a coordinate system. The concepts of point, direction, length and displacement are required for measurements and locations in space. For example, a point is just an exact indication of a location in space, requiring no space at all for itself. Length is concept for describing the size of a body quantitatively by comparing it with a second body of known size. For two-dimensional problems, two independent co-ordinates are needed. For three-dimensional problems, three independent co-ordinates are required.

Mass Mass is the quantity of matter in a body. Matter refers to the substance of which physical bodies are composed. Each body is made up of atoms and molecules. Mass can also be regarded as a measure of the inertia of a body, which is its resistance to a change of motion.

Time Time is the measure of sequence of events. Time is related to the concepts of before, after and simultaneous occurrence of two or more events. Time is a basic quantity in dynamics and it is not directly involved in the analysis of statics problems.

Force Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction. Force is a vector quantity and its properties are discussed in detail in Chapter 2. In Newtonian mechanics, space, time and mass are absolute concepts, i.e., independent of each other. The concept of force is related to the mass of the body with Newton's Second law.

System of Forces

When several forces of various magnitudes and directions act upon a body, they are said to constitute a system of forces. The system of forces may be classified according to the orientation of the lines of action of the forces as follows: Force Systems in Plane: System of forces consists of a set of forces with their lines of action lying in the same plane. Force Systems in Space: System of forces consists of a set of forces with their lines of action lying in the space.

Both force systems in plane and force systems in space can further be classified into (i) concurrent force system, (ii) parallel force systems and (iii) non-concurrent or general force system. The classification of force systems is shown as in Fig. 1.1. In general, we can have six types of force systems.

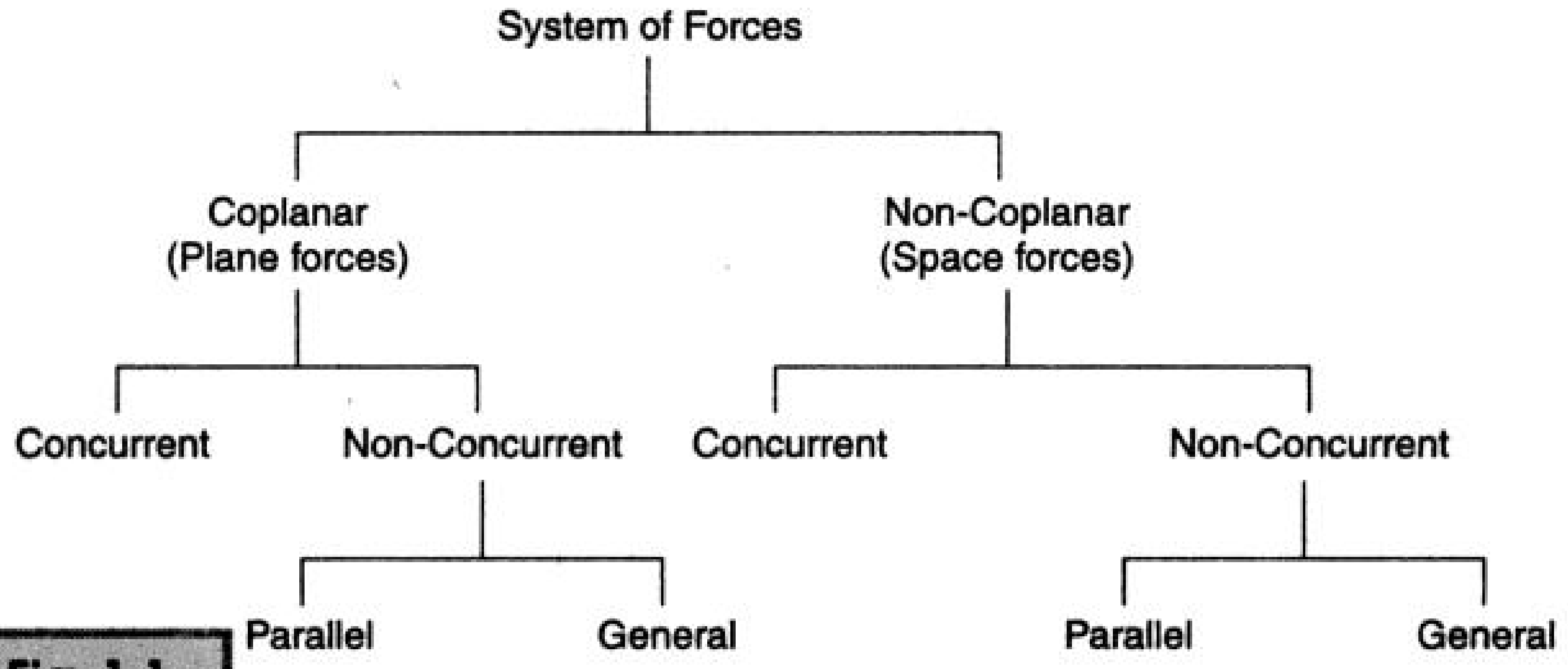


Fig. 1.1

Concurrent Force System in a Plane In this system, the lines of action of all forces pass through a single point and forces lie in the same plane (Fig. 1.2).

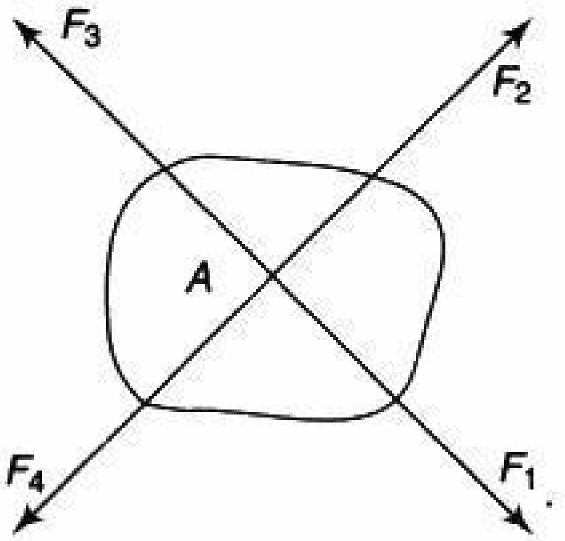
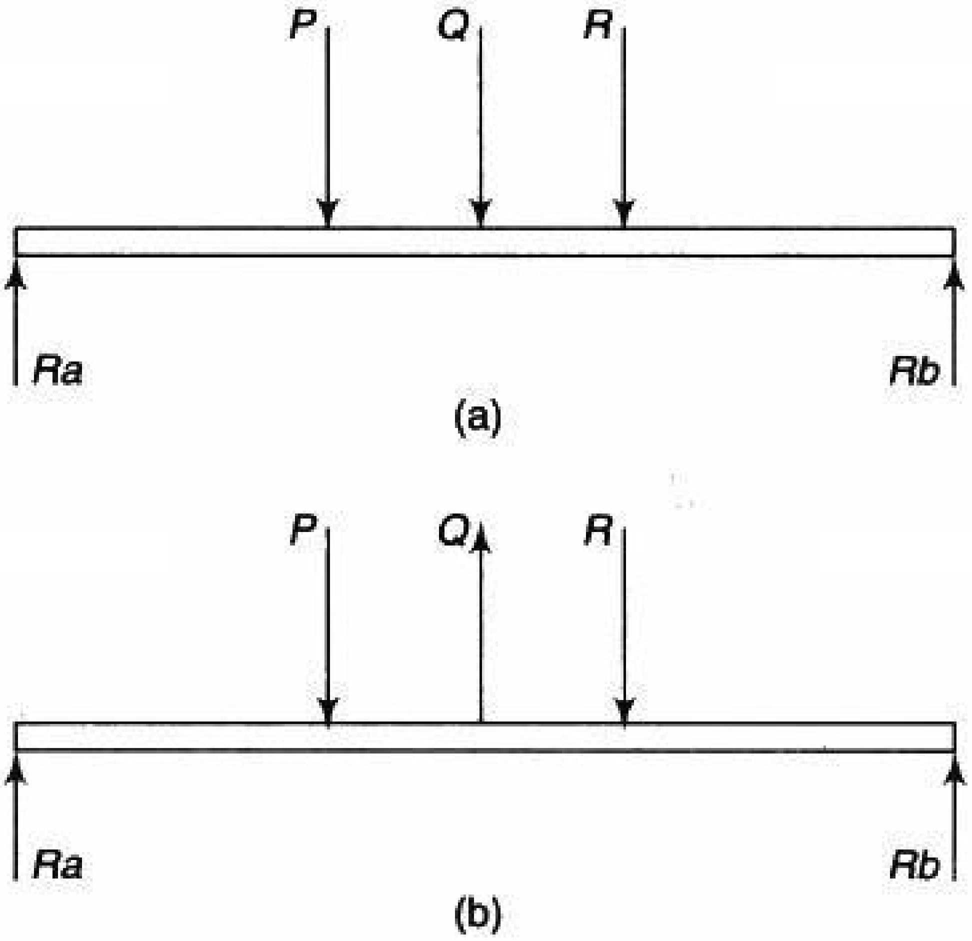


Fig. 1.2

Parallel Force System in a Plane In this system, the lines of action of all forces lie in the same plane and are parallel to each other (Fig. 1.3).



(a)

(b)

Fig. 1.3

General Force System in a Plane The lines of action of these forces lie in the same plane but they are neither parallel nor concurrent (Fig. 1.4).

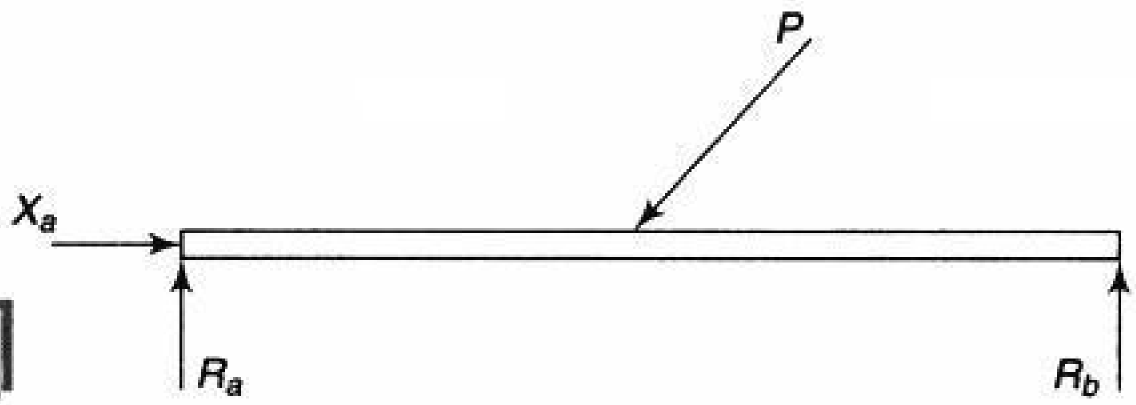


Fig. 1.4

Concurrent Force System in Space The lines of action of all forces pass through a single point but not lie in the same plane. Tripod carrying a camera as shown in Fig. 1.5 is an example.

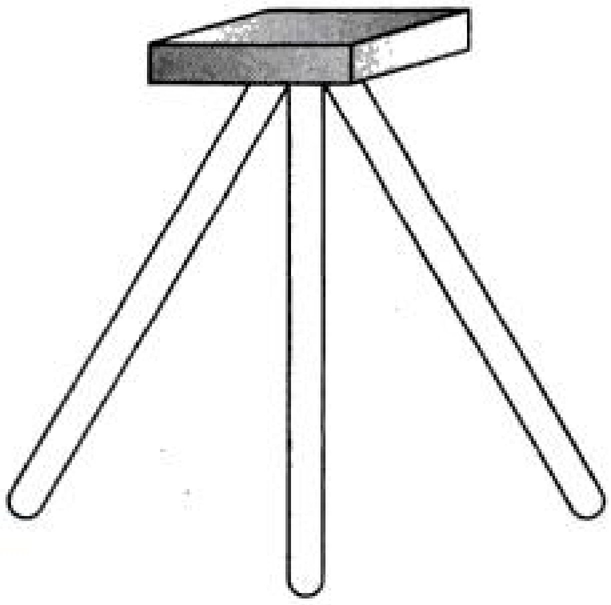


Fig. 1.5

Parallel Force System in Space The lines of action of all forces are parallel to each other, but not lie in the same plane (Fig. 1.6).

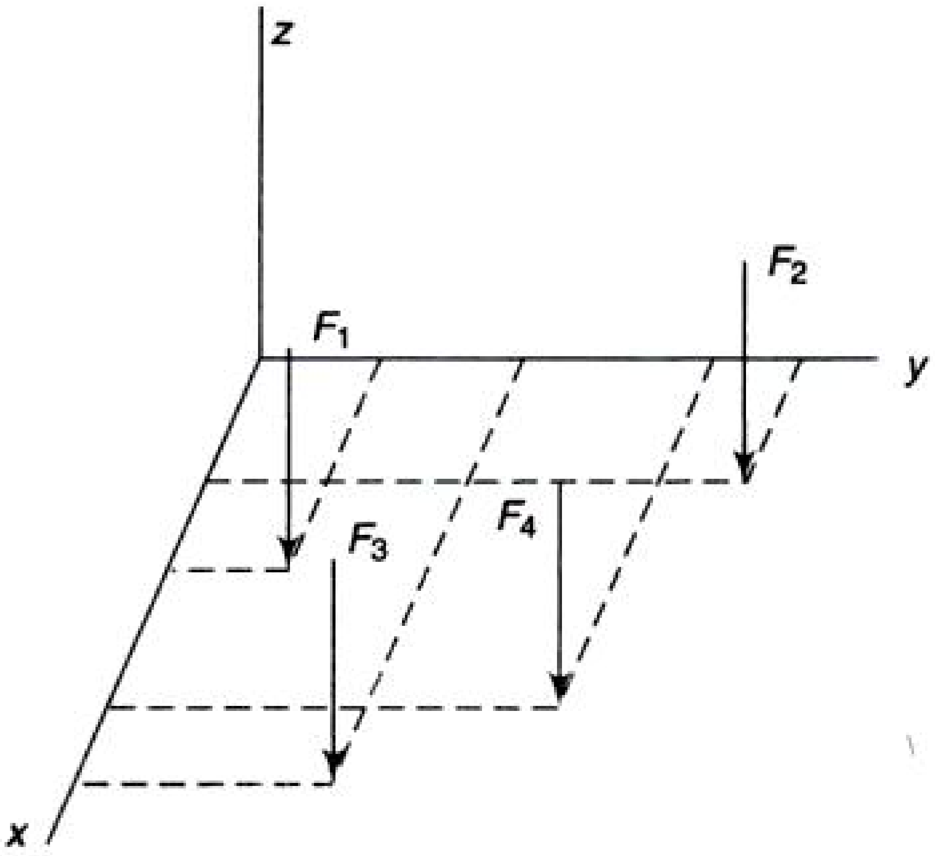


Fig. 1.6

General Force System in Space The lines of action of these forces do not lie in the same plane and they are neither parallel nor concurrent (Fig. 1.7).

Concurrent force systems can act on a particle or a rigid body. Parallel and

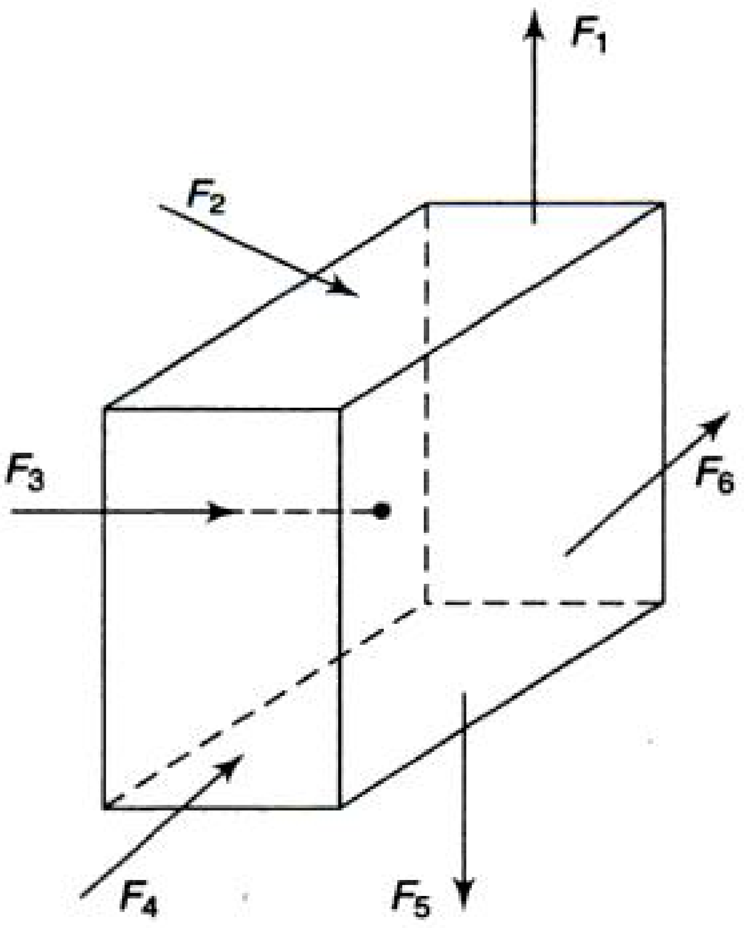


Fig. 1.7

general force systems can act only on a system of particles, a rigid body or a system of rigid bodies. In this book, we will study statics according to force systems.

Important Terms and Concepts

Engineeering
Mechanics
Scientists
Engineeers
Design
Statics
Dynamics
Kinematics
Kinetics
Idealizations
Continuum
Particle
System of particles
Rigid body
Basic concepts
Mass
Time
Space
Force
Specifications of a force
System of forces
Kinds of forces

SUMMARY

- Engineering mechanics may be defined as the science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Depending upon the nature of the problems treated, mechanics is divided into statics and dynamics.
- Statics is the study of the forces and the conditions of equilibrium of material bodies subjected to the action of forces.
- Dynamics is the study of motion of rigid bodies and their correlation with the forces causing them. Dynamics is divided into kinematics and kinetics.
- Kinematics deals with the space–time relationship of a given motion of a body and not at all with the forces that cause the motion.
- Kinetics studies the laws of motion of material bodies under the action of forces.
- Mathematical models or idealizations are used in mechanics to simplify the application of theory.
- Continuum may be defined as a continuous distribution of matter with no voids or empty spaces.
- A particle is a point mass or a material point.
- A system of particles is an idealization of point masses. A system of particles is constituted, when two or more bodies are represented by particles and are dealt with together.
- A rigid body is the one in which the distance between any two arbitrary points is invariant.
- The basic concepts used in mechanics are space, time, mass and force. These basic concepts cannot be truly defined but are developed for axiomatic thinking and mutual understanding.
- Space refers to the geometric region occupied by bodies.
- Mass is the quantity of matter in a body.
- Time is the measure of sequence of events.
- Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction. Force is a vector quantity.
- In Newtonian mechanics, space, time and mass are absolute concepts, i.e., independent of each other. The concept of force is related to the mass of the body with Newton's Second law.
- When several forces of various magnitudes and directions act upon a body, they are said to constitute a system of forces.

PRACTICE SET 1.1

Review Questions

1. Define engineering mechanics.
2. State and explain the idealisations in mechanics.
3. What are the basic concepts used in engineering mechanics?
4. Explain the term 'system of forces'.
5. Distinguish between statics and dynamics.

Objective Questions

1. Select the incorrect statement.
   (a) Statics is the study of forces and the conditions of equilibrium of material bodies subjected to the action of forces.
   (b) Dynamics is the study of motion of bodies and their correlation with the forces causing them.
   (c) Kinematics deals with the space–time relationship of a given motion of a body.
   (d) Kinematics deals with the relationship between the forces and the resulting motion.

   [*Ans.* (d)]
2. Select the incorrect statement.
   (a) Continuum may be defined as a continuous distribution of matter with no voids.
   (b) A rigid body is the one in which the distance between any two arbitrary points is variant.
   (c) A particle is a point mass.
   (d) A system of particles is an idealization of point masses.

   [*Ans.* (b)]
3. While studying the planetary motion, sun and planets are considered as
   (a) Deformable body (b) Rigid body
   (c) Particles (d) None of the above.

   [*Ans.* (c)]
4. Which of the following is the basic concept of mechanics?
   (c) Charge (b) Power
   (c) Force (d) Energy

   [*Ans.* (c)]

1.2 UNITS AND DIMENSIONS

Unit is defined as the numerical standard used to measure the qualitative dimension of a physical quantity. When mass or force, length and time are accepted as basic quantities, then all other quantities are secondary or derived quantities in terms of these basic quantities.

Fundamental Units and Derived Units

The units in which the fundamental quantities are measured are called as fundamental or basic units. The three primary units basic to mechanics are length, time and mass.

The derived units are the units of derived physical quantities, which are expressed in terms of the fundamental units. Examples: Area, Volume, Force, Velocity, etc.

Coherent System of Units

A coherent system of units is the one in which the units of derived quantities are obtained as multiples or sub-multiples of certain basic units.

SI Units

The International System of Units, abbreviates SI (from the French, Systeme International d'Unites), has been accepted throughout the world and is a modern version of the metric system. In SI units, length in meters (m), mass in kilograms (kg) and time in seconds (s) are selected as the base units and force in newtons (N) is derived from Newton's Second law. One newton is the force required to give a mass of 1 kg an acceleration of 1 m/s^2.

Primary standards for the measurements of length, mass and time are as follows.

Length The meter is defined as 1 650 763.73 wavelengths of a certain radiation of the krypton-86 atom at 15°C and 76 cm of mercury.

Mass The kilogram is defined as the mass of a platinum Iridium cylinder of diameter equal to its height kept at the International Bureau of weights and measures near Paris France.

Time The second is defined as the duration of 9 192 631 770 periods of the radiation of a certain state of the cesium-133 atom.

Note ➻ SI units are absolute system of units (units of mass, length and time are basic units), which are independent of the location where the measurements are made.

The SI units consist of 7 base units, 2 supplementary units and a number of derived units. While studying engineering mechanics, it is sufficient if one knows about the three basic units given below in Table 1.1. The SI derived units with new names are given in Table 1.2. SI units of some common physical quantities are summarized in Table 1.3. When a numerical quantity is either very large or very small, units used to define its size may be modified by using a prefix. Some of the multiple or submultiples used in the SI system are shown in Table 1.4.

Table 1.1

Quantity	*SI units*	*Symbol*
Mass	Kilogram	kg
Length	Meter	m
Time	Second	s

Table 1.2

Derived unit	*Symbol*	*Physical quantity*
Newton	N = kg m/s^2	Force
Joule	J = N m = kg m^2/s^2	Energy, Work, Heat
Watt	W = J/s = N m/s = kg m^2/s^2	Power
Pascal	Pa = N/m^2 = kg m/s^2	Pressure, Stress
Hertz	Hz = s^{-1}	Frequency

Table 1.3

Physical quantity	*Unit*	*Symbol*
Acceleration	Metre/second^2	m/s^2
Angular acceleration	Radian/second^2	rad/s^2
Angular displacement	Radian	rad
Angular momentum	Kilogram metre^2/second	kg m^2/s
Angular velocity	Radian/second	rad/s
Area	Square metre	m^2
Couple, moment	Newton metre	N m
Density	Kilogram/metre^3	kg/m^3
Displacement	Metre	m
Energy	Joule	J
Force	Newton	N
Frequency	Per second	Hz
Length	Metre	m
Mass	Kilogram	kg
Momentum	Kilogram metre/second	kg m/s (=N s)
Moment of inertia of mass	Kilogram metre^2	kg m^2
Plane angle	Radian	rad
Power	Watt	W
Pressure	Pascal	Pa
Speed	Metre/second	m/s
Time	Second	s

Table 1.4

Multiplication factor	*Prefix*	*Symbol*
10^{12}	Tera	T
10^9	Giga	G
10^6	Mega	M
10^3	Kilo	k
10^{-3}	Milli	m
10^{-6}	Micro	μ
10^{-9}	Nano	n
10^{-12}	Pico	p

Rules for Use of SI Symbols

The following rules are given for the proper use of the various symbols in SI units.

1. A symbol is never written with a plural "s".
2. Symbols are always written in lowercase letters except the symbols named after an individual e.g., N and J.

3. Kilogram is written as kg and not as kgm, kgf, etc. Similarly, second as s, not sec. or sec, etc. No full stops, dots or dashes should be used. For example, moment is in N m, not N.m, N-m, etc.
4. It is permissible that one space be left between any two unit symbols, e.g., kg m^2/s, m s.
5. No space be left after a multiple or submultiple symbol, e.g., kJ/kg K.
6. Always leave a space between the number and the unit symbol, e.g., 3 m, 1500 N.
7. For numbers less than unity, zero must be put on the left of the decimal, e.g., 0.30 m. For large numbers exceeding five figures, one space after every three digits counting from the right end must be left blank without any commas, e.g., 1 500 375 is the correct way of writing the number.
8. The exponential power represented for a unit having a prefix refers to both the unit and its prefix. For example, mm^2 = (mm)^2 = mm . mm.
9. Represent the numbers in terms of the base or derived units by converting all prefixes to powers of, while performing calculations. The final result should then be expressed using a single prefix.
10. In general, avoid the use of a prefix in the denominator of composite units. Exception for this is the base unit, kilogram. For example, do not write N/μm, but rather MN/m.
11. Compound prefixes should not be used, e.g., GμN (giga-micro-newton) should be expressed as kN since 1 GμN = 1 (10^9) (10^{-6}) N = 1 (10^3) N = 1 kN.

Dimensions

Dimensional analysis deals with dimensions of quantities (Table 1.5).

Table 1.5

Base unit	*Dimension*
Mass	M
Length	L
Time	T
Temperature	K
Electric current	A
Luminous intensity	Cd
Amount of substance	mol

Dimensional Formula Dimensional formula is a formula in which the given physical quantity is expressed in terms of the fundamental quantities raised to suitable power.

In order to completely define a physical quantity, the following are to be known: (1) the unit of the quantity, (2) the number of times the unit contained in that quantity, i.e., the numerical value. Area is represented by L^2 and area has 2 dimensions in length. Volume is represented by L^3 and has 3 dimensions in length.

The dimensional formula of some of the derived quantities are given in Table 1.6.

Table 1.6

S.No.	Physical quantity	Expression	Dimensional formula
1.	Displacement (S)	Distance	L
2.	Velocity (V)	Distance/Time	LT^{-1}
3.	Acceleration (a)	Velocity/Time	LT^{-2}
4.	Force (F)	Mass × Acceleration	MLT^{-2}
5.	Momentum	Mass × Velocity	MLT^{-1}
6.	Impulse	Force × Time	MLT^{-1}
7.	Work or Energy	Force × Displacement	ML^2T^2
8.	Power	Work/Time	ML^2T^3
9.	Pressure	Force/Area	$ML^{-1}T^1$
10.	Frequency	No. of vibrations/Time	T^{-1}
11.	Angular Velocity	Angle/Time	T^{-1}

Uses of Dimensional Formula The dimensional formula is used

(i) to check the correctness of a given equation,

(ii) to derive equations for physical quantites involved in the problem.

Dimensionless Quantities Dimensionless quantities are the quantities that are the ratio of two quantities having the same dimensional formula.

Law of Dimensional Homogeneity

The law of dimensional homogeneity states that all equations which describe the physical processes must be dimensionally homogeneous. In other words, no equation in which the seperate terms have different dimensions can be physically valid. Dimensionally homogenous equations are also said to be dimensionally correct. In a dimensionally correct, all the terms on the left and right side will have the same dimensions.

Law of dimensional homogeity is used (a) to verify whether the given equation is dimensionally correct or not (b) to find the dimensions of certain terms in a dimensionally homogeneous equation.

Important Terms and Concepts

Unit
SI units
Law of dimensional homogeneity
Fundamental unit
Rules for use of SI units
Derived unit
Dimensions
Coherent system of units
Dimensional formula

SUMMARY

- Unit is defined as the numerical standard used to measure the qualitative dimension of a physical quantity.
- The units in which the fundamental quantities are measured are called as fundamental or basic units. The three primary units basic to mechanics are length, time and mass.
- The derived units are the units of derived physical quantities, which are expressed in terms of the fundamental units (examples: Area, Volume, Force, Velocity, etc.).

- The International System of Units, abbreviates SI (from the French, Systeme International d'Unites), has been accepted throughout the world and is a modern version of the metric system. In SI units, length in meters (m), mass in kilograms (kg) and time in seconds (s) are selected as the base units.
- Dimensional formula is a formula in which the given physical quantity is expressed in terms of the fundamental quantities raised to suitable power.
- The law of dimensional homogeneity states that all equations which describe the physical processes must be dimensionally homogeneous.

PRACTICE SET 1.2

Review Questions

1. Define unit.
2. Differentiate between fundamental units and derived units.
3. Explain SI units.
4. Define dimensional formula.
5. What are the uses of dimensional formula?
6. State the law of dimensional homogeneity.

Objective Questions

1. Joule is the unit of

   (a) Power (b) Moment
   (c) Momentum (d) Work

   [*Ans.* (d)]

2. In SI units, the units of force and power are respectively

   (a) Newton and Watt (b) Newton and Joule
   (c) Newton and Pascal (d) Newton and Hertz

   [*Ans.* (a)]

3. Which of the following is dimensionless quantity?

   (a) Force (b) Velocity
   (c) Pure number (d) Frequency

   [*Ans.* (c)]

4. The dimension of linear momentum in MLT system is

   (a) MLT (b) MLT^{-1}
   (c) MT (d) LT^{-1}

   [*Ans.* (b)]

5. The dimension of pressure in MLT system is

   (a) $ML^{-1}T^{-1}$ (b) $ML^{-1}T^{2}$
   (c) $ML^{-1}T^{-2}$ (d) $ML^{2}T^{2}$

   [*Ans.* (c)]

6. Consider the following statements.

   I. The dimensional formula is used to derive the equations for physical quantities involved in the problem.
   II. Dimensionless quantities are the quantities that are the ratio of two quantities having the same dimensional formula.
   III. The law of dimensional homogeneity states that all equations which describe theprocesses must be dimensionally homogeneous.

   Out of these statements.

   (a) Only I is correct. (b) I and II are correct
   (c) All are incorrect (d) All are correct.

   [*Ans.* (d)]

1.3 METHOD OF PROBLEM SOLUTION AND THE ACCURACY OF SOLUTIONS

The solution of a problem in engineering mechanics usually consists of three steps: (1) the reduction of a complex physical problem to such a state of idealization that it can be expressed algebraically or geometrically; (2) the solution of this purely mathematical problem; and (3) the interpretation of the results of the solution in terms of the given physical problem.

The mastery of the principles of engineering mechanics will be reflected in the ability to formulate and solve problems. There is no simple method for teaching problem-solving skills. To develop the analytical skills that are so necessary for success in engineering, a considerable amount of practice in solving problems is necessary. For this reason, a relatively large number of examples and practice set are given throughout the text. An effective method of attack on statics problems is essential. Each solution should proceed with a logical sequence of steps from hypothesis to conclusion. Its representation should include a clear statement of the following parts, each clearly identified :

1. Given data: After carefully reading the problem statement, list all the data provided.
2. Results desired or find: State precisely the information that is to be determined.
3. Necessary diagrams: If a figure is required, sketch it neatly and approximately to scale.
4. Calculations or solution: Solve the problem, showing all the steps that you used in the analysis. Work neatly so that others can easily follow your work.
5. Answers and conclusions or validate: Many times, an invalid solution can be uncovered by simply asking yourself, "Does the answer make sense?"

The accuracy of the solution of a problem depends upon the two items: (1) the accuracy of the given data and (2) the accuracy of the computations performed. The data given in a problem should be assumed known with a comparable degree of accuracy. The number of significant figures in an answer should be no greater than the number of figures, which can be justified by the accuracy of the given data. A practical rule is to use 4 figures to record numbers beginning with a "1" and 3 figures in all other cases.

2

Concurrent Forces in a Plane

2.1 PRINCIPLES OF STATICS

Statics deals with the conditions of equilibrium of bodies acted upon by forces and is one of the oldest branches of science. Some of its fundamental principles date back to the Egyptians and Babylonians who used them in the solution of problems encountered in building the famous pyramids and old temples.[1] The earliest writings on the subject were left by Archimedes (287–212 B.C.), who formulated the laws of equilibrium of forces acting on a lever and also some principles of hydrostatics. However, the principles from which the subject in its present form is developed were not fully stated until the later part of the seventeenth century and are mainly the work of Stevinus, Varignon and Newton, who were the first to use the principle of the parallelogram of forces.

Statics deals with the conditions of equilibrium of bodies acted upon by forces.

Rigid Body

We shall be mostly concerned in this book with problems involving the equilibrium of rigid bodies. Physical bodies, such as we have to deal with in the design of engineering structures and machine parts, are never absolutely rigid but deform slightly under the action of loads which they have to carry. Consider, for example, the lever shown in Fig. 2.1(a). Under the action of the two equal weights at the ends, the bar bends slightly over the fulcrum and the distance of each weight from the fulcrum is decreased by a very small amount. In discussing the equilibrium of the lever (equal weights at equal distances from the fulcrum are in equilibrium), we may safely ignore this deformation and assume that the lever is a rigid body which remains straight, as shown in Fig. 2.1(b). That is, we assume that the distance of each weight from the fulcrum is half the length of the bar. This illustrates the significance of the assumption of rigid bodies in dealing with static equilibrium.

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[1] For historical data regarding the development of statics, see Ernst Mach, "Science of Mechanics," Open Court Publishing Company, Chicago, 1902.

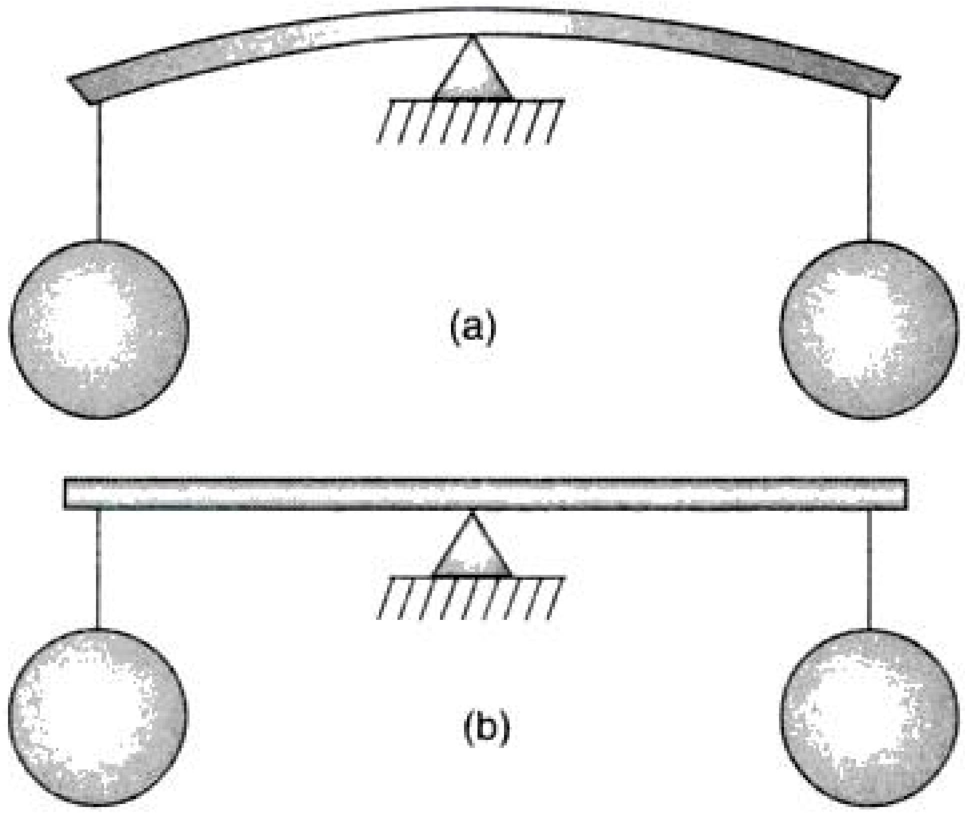


Fig. 2.1

*A **rigid body** is defined as a definite quantity of matter the parts of which are fixed in position relative to one another. The physical bodies are never absolutely rigid but deform slightly under the action of loads, which they have to carry. If the deformation is negligible when compared with the size of the body, it is assumed to be rigid.*

If we are interested in the *strength* of the lever in Fig. 2.1, or the amount of sag, the deformation represented by the bent form becomes important and must be taken into account. Problems in which the effect of small deformations of physical bodies must be taken into account are generally treated in books on *strength of materials* or *theory of elasticity*.

Problems dealing with the conditions of equilibrium of nonrigid bodies, such as liquids and gases, are usually treated in books on *fluid mechanics*. These will also not be considered here, except in so far as they may be involved in determining the pressure or loading exerted on rigid bodies that we do have under consideration.

Force

For the investigation of problems of statics we must introduce the concept of *force*, which may be defined as any action that tends to change the state of rest or motion of a body to which it is applied. A force or motion acting on a rigid body produces one or both of the following effects: (i) linear displacement, (ii) angular (rotating) displacement. These effects essentially result in a change in the state of rest or motion of a body.

***Force** may be defined as any action that tends to change the state of rest or motion of a body to which it is applied.*

There are many kinds of force, such as gravity force with which we are all familiar and the simple push or pull that we can exert upon a body with our hands. Other examples of force are the gravitational attraction between the sun and planets, the tractive effort of a locomotive, the force of magnetic attraction, steam or gas pressure in a cylinder, wind pressure, atmospheric pressure and frictional resistance between contiguous surfaces.

Weight is the force of gravitational attraction of the earth on a body.

Hydrostatic pressure: When a body such as a dam impounds water, the water exerts a force on th e imppounding bodywhich is distributed on the area of its contact with the body. This is known as hydrostatic pressure.

Gas pressure: Gas confined under pressure in a container exerts pressure over the entire area of the container, called gas pressure.

Earth pressure: Earth piled up against a wall exerts pressure on it called the earth pressure.

Wind pressure: Bodies exposed to wind are subjected to a force distributed over their exposed area, called the wind pressure.

The pull of gravity is one of the most common examples of force with which we shall have to deal. Given a ball that hangs by a string [Fig. 2.2(a)], we say that the ball pulls on the string with a force *W* equal to its weight. This force is applied to the string at point *B* and acts vertically downward.

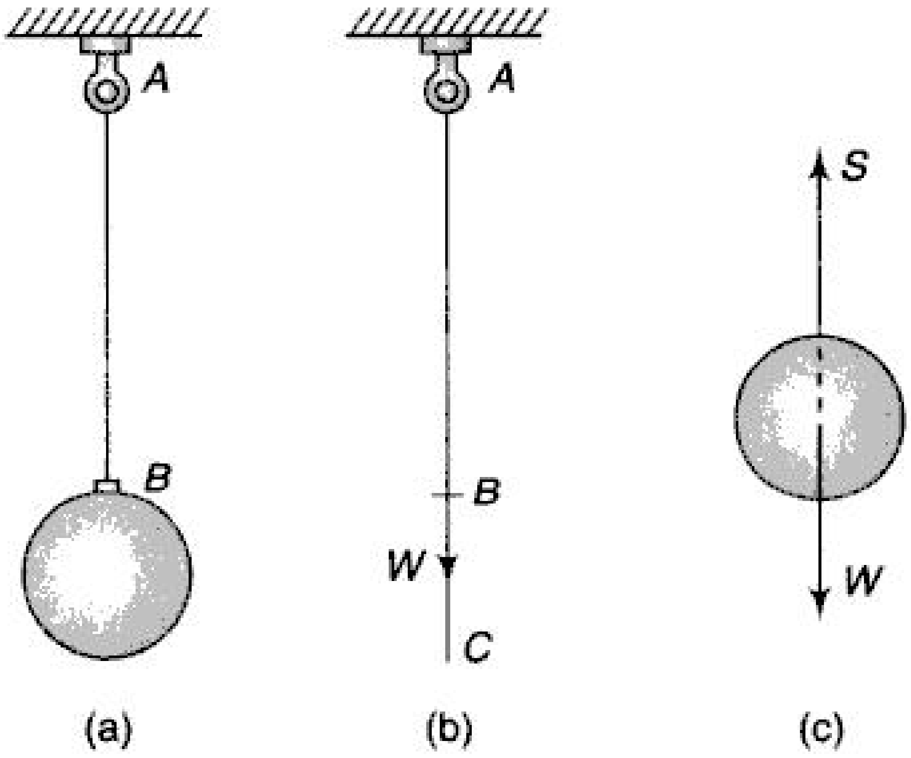


Fig. 2.2

From the above example, we see that for the complete definition of a force we must know (1) its magnitude, (2) its point of application, and (3) its direction. These three quantities which completely define the force are called its specifications or characteristics.

The specifications or characteristics of a force are (1) its magnitude, (2) its point of application, and (3) its direction.

The SI units used by engineers to measure the magnitude of a force are the Newton (N) and its multiple the kilonewton (kN), equal to 1000 N. The Newton is a derived unit. One Newton is defined as the force, which gives an acceleration of 1 m/s^2 to a mass of 1 kg.

The point of application of a force, acting upon a body is that point in the body at which the force can be assumed to be concentrated. Physically, it will be impossible to concentrate a force at a single point; i.e., every force must have some finite area or volume over which its action is distributed. For example, the force *W* exerted by the ball upon the string *AB* in Fig. 2.2 is in reality distributed over the small cross-sectional area of the string. Likewise, the gravity force, which the earth exerts on the ball, is distributed throughout the volume of the ball. However, we often find it convenient to think of such distributed force as being concentrated at a single point of application wherever this can be done without sensibly changing the effect of the force on the conditions of equilibrium. In the case of gravity force distributed throughout the volume of a body, the point of application at which the total weight can be assumed to be concentrated is called the *center of gravity* of the body.

Concentrated force or a point load is a force acting over a very small area.

Distributed force is a force distributed over a length or an area or a volume.

The direction of a force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied. This line is called the *line of action* of the force. The force of gravity, for example, is always directed vertically downward. Again, in the case of a force exerted upon a body by a flexible string, the string defines the line of action of the force. Thus the string *AB* in Fig. 2.2 pulls vertically downward on the hook at *A*.

Any quantity, such as force, that possesses direction as well as magnitude is called a *vector quantity* and can be represented graphically by a segment of a straight line, called a *vector.* For example, in Fig. 2.2(b), we can represent the force that the ball exerts on the string by the straight-line segment *BC*, the length of which, to some convenient scale, shows the magnitude of the force and the vertical downward direction of which, indicated by the arrow, shows the direction of the force. Point *B* is called the *beginning* of the vector; and point *C*, the *end*. Either the beginning or the end of a vector may be used to indicate the point of application of the force. With the beginning and the end of a vector indicated by letters [as *B* and *C* in Fig. 2.2(b)] we shall designate the vector by the symbol $\overline{BC}$, which defines it specifically as acting from *B* toward *C*.

Representation of Force Graphically a force may be represented by the segment of a straight line. The straight line represents the line of action of the force 1 kN and its length represents its magnitude. The direction (or sense) of the force is indicated by placing an arrow head on this straight line (Fig. 2.3). Either the head or the tail may be used to indicate the point of application of a force. Note that all the forces involved must be represented consistently.

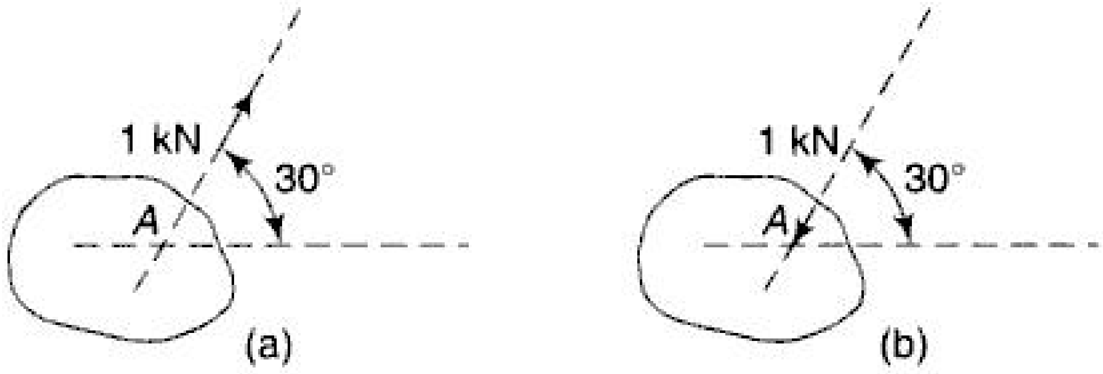
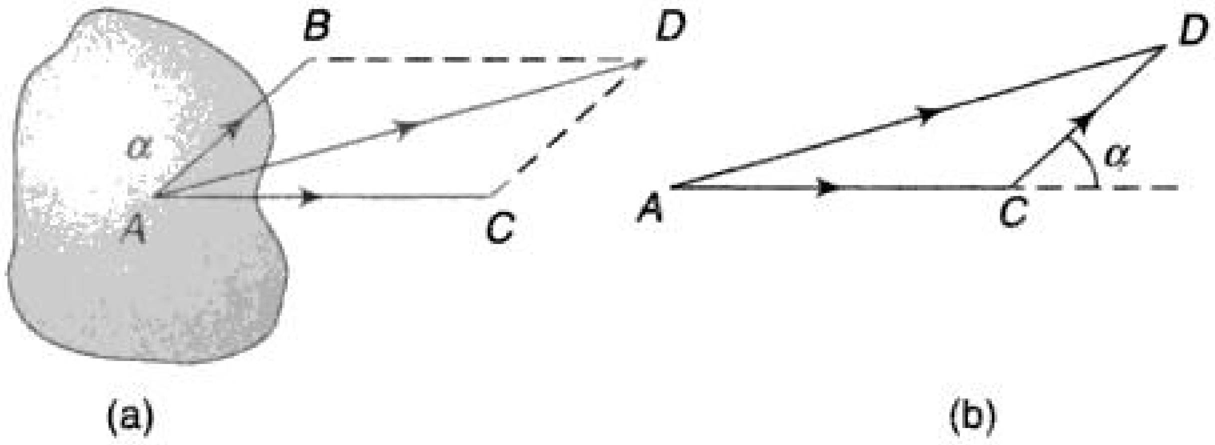


Fig. 2.3

Parallelogram of Forces When several forces of various magnitudes and directions act upon a body, they are said to constitute a *system of forces*. The general problem of statics consists of finding the conditions that such a system must satisfy in order to have equilibrium of the body. The various methods of solution of this problem are based on several axioms, called the *principles of statics*. We begin with the principle of the *parallelogram of forces*, first employed indirectly by Stevinus in 1586 and finally formulated by Varignon and Newton in 1687.

Composition of Two Forces The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces.* Here we will discuss the reduction of a given system of forces, i.e., two forces to the simplest system that will be its equivalent, i.e., resultant with the help of parallelogram law.

Parallelogram Law: *If two forces, represented by vectors* $\overline{AB}$ *and* $\overline{AC}$*, acting under an angle* α [Fig. 2.4(a)] *are applied to a body at point* A, *their action is equivalent to the action of one force, represented by the vector* $\overline{AD}$ *, obtained as the diagonal of the parallelogram constructed on the vectors* $\overline{AB}$ *and* $\overline{AC}$ *and directed as shown in the figure.*



(a) (b)

Fig. 2.4

The force $\overline{AD}$ is called the resultant of the two forces $\overline{AB}$ and $\overline{AC}$. The forces $\overline{AB}$ and $\overline{AC}$ are called *components* of the force $\overline{AD}$. Thus a force is equivalent to its components, and vice versa.

Instead of constructing the parallelogram of forces, the resultant can be obtained also by constructing the triangle *ACD*, as shown in Fig. 2.4(b). Here we take the vector $\overline{AC}$ and from its end *C* draw the vector $\overline{CD}$, equal and parallel to the vector $\overline{AB}$. Then the third side $\overline{AD}$ of the triangle gives the resultant, being directed from *A*, the beginning of the vector $\overline{AC}$, to *D*, the end of the vector $\overline{CD}$. The vector $\overline{AD}$, when obtained in this way, is called the *geometric sum of* the vectors $\overline{AC}$ and $\overline{CD}$. Thus, the magnitude and direction of the resultant of two forces, applied to a body at point *A*, may be obtained as the geometric sum of the two vectors representing these forces. Its point of application, of course, is also point *A*. Since the vectors in Fig. 2.4(b) do not show the points of application of the forces that they represent, they are called *free vectors.* The triangle *ACD* is called a *triangle of forces.*

If two forces $\overline{AB}$ and $\overline{AC}$ act under a very small angle [Fig. 2.5(a)], the triangle of forces [Fig. 2.5(b)] becomes very narrow and we conclude that, in the limiting case, where the two forces act along the same line and in the same direction, their resultant is equal to the sum of the forces and acts in the same direction. In the same manner it can be shown that, if two forces act along the same line in opposite directions, their resultant is equal to the difference between the forces and acts in the direction of the larger force. By taking one direction as positive and the other as negative along the common line of action of two forces and considering the forces themselves as positive or negative accordingly, we conclude that the resultant of two *collinear forces* is equal to their *algebraic sum.*

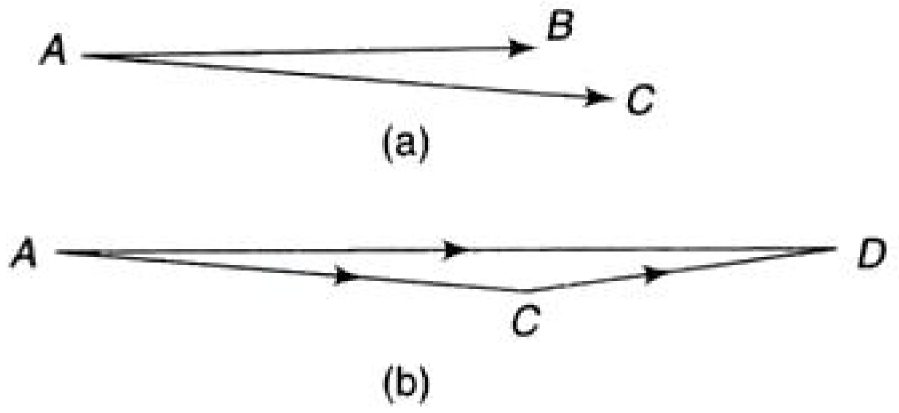


Fig. 2.5

Analytical Method If two given forces P and Q, acting under the angle α, are applied to a body at A, we will now find analytically the formulae for calculating the magnitude of their resultant R and the angles β and γ which its line of action makes with those of the given forces. Also give the formulae for the resultant in the special cases, where $\alpha = 0°$ and $\alpha = 180°$ and $P > Q$ and $\alpha = 90°$.

Figure 2.6(a) shows the parallelogram of forces constructed in the usual manner, while Fig. 2.6(b) shows the triangle of forces obtained by the geometric addition of their free vectors. From the triangle of forces we find

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} \qquad \text{(a)}$$

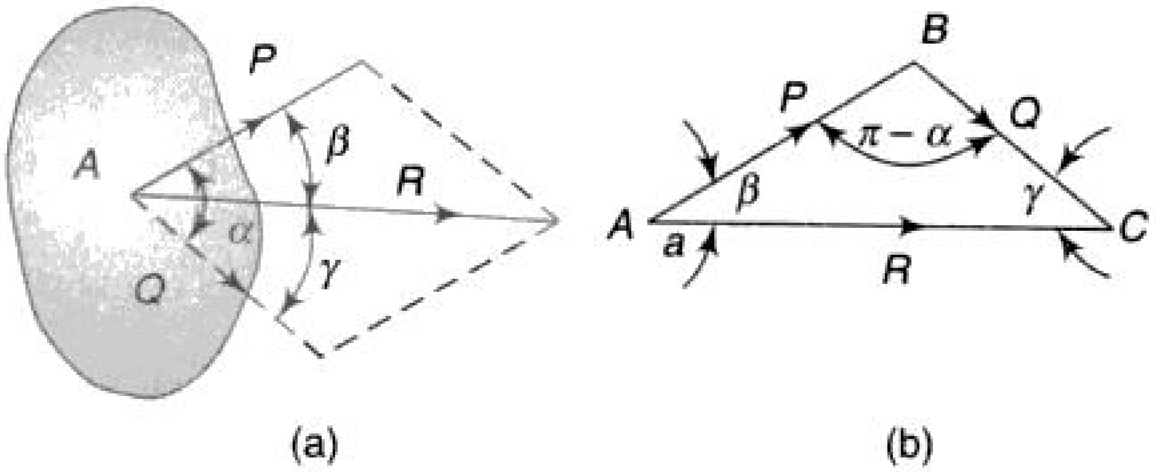


Fig. 2.6

The magnitude of the resultant R being known from Eq. (a), we may determine the angles β and γ by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha \qquad \sin\gamma = \frac{P}{R}\sin\alpha \qquad \text{(b)}$$

It is sometimes convenient to use these formulas for determining the resultant instead of making an accurate construction, to scale, of the triangle of forces.

For the special case (i) $\alpha = 0°$,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 0}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ(1)} = \sqrt{(P+Q)^2} = P + Q$$

$$\therefore \qquad R = P + Q$$

For the special case (ii) $\alpha = 180°$,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos 180}$$

$$R = \sqrt{P^2 + Q^2 + 2PQ(-1)} = \sqrt{(P - Q)^2} = P - Q$$

$$\therefore \qquad R = P - Q$$

In the special cases, the resultant of two forces became algebraic sum of the two forces.

Case (iii) if $\alpha = 90°$, i.e., rectangular components, then the resultant is given by the equation

$$R = \sqrt{P^2 + Q^2 + 2PQ \cos 90°} = \sqrt{P^2 + Q^2}$$

and $$\tan \gamma = \frac{P}{Q}$$

where γ = the angle between resultant R and the force Q.

Resolution of a Force The replacement of a single force by several components, which will be equivalent in action to the given force, is called the problem of *resolution of a force*. The case in which a single force is to be replaced by two components is the one most commonly encountered. By using the parallelogram law, we can resolve a given force R into any two components P and Q intersecting at a point on its line of action. We shall discuss two possible cases.

1. The directions of both components are given; their magnitudes, to be determined. Imagine, for example, that the force R, represented by the vector $\overline{AB}$ [Fig. 2.7(a)], is to be resolved into two components acting along the lines AC' and AD'. We proceed by drawing from point B the dotted lines BC and BD, parallel to the given lines of action of the desired components. The points C and D, where these lines intersect the given lines of action of the components, determine the vectors $\overline{AC}$ and $\overline{AD}$ which completely define the two components P and Q.

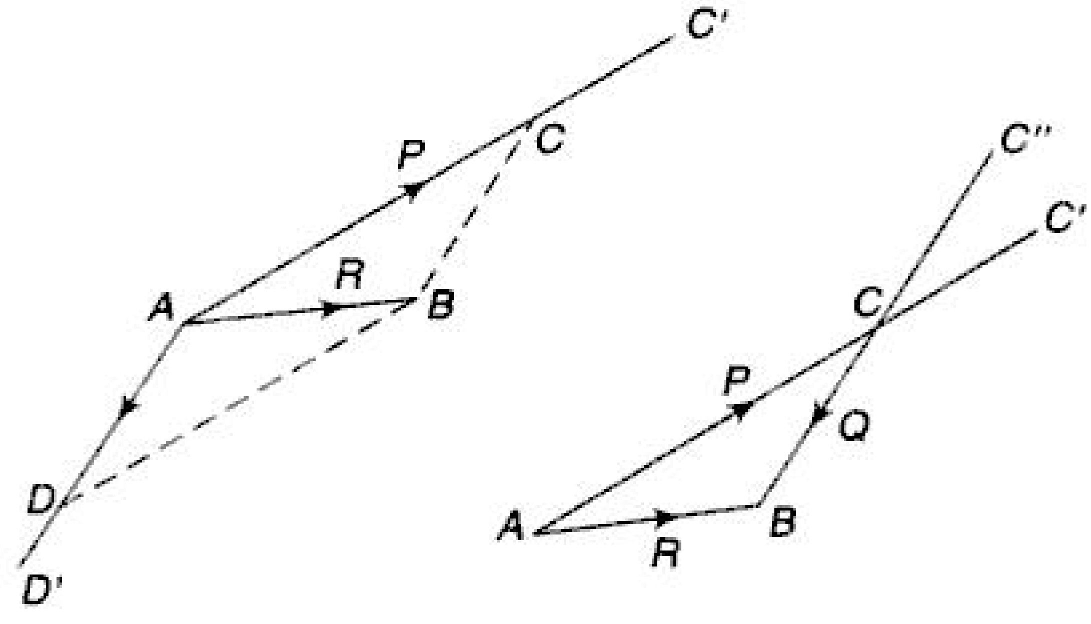


Fig. 2.7 (a) (b)

We can obtain the same result by using the triangle of force ABC as shown in Fig. 2.7(b). Here the lines AC' and BC'', parallel to the given lines of action of the components, are extended from the beginning A and the end B of the vector $\overline{AB}$ representing the given force R and their point of intersection C determines the vectors $\overline{AC}$ and $\overline{CB}$, representing the components P and Q. These components, applied at any point on the line of action of the force R, will be its equivalent. In the particular case where the two components act at right angles to each other, they are called *rectangular components*.

2. Both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined. For example, imagine that the force R, represented by the vector $\overline{AB}$, and the component P, represented by the vector $\overline{AC}$ [Fig. 2.7(a)], have been given. Laying out these two vectors as shown in Fig. 2.7(b), the magnitude and direction of the other component Q are given by the vector $\overline{CB}$, obtained by joining the ends C and B of the two given vectors.

Analytical Method

1. The directions of both components are given; their magnitudes, to be determined. Imagine, for example, that the force R [Fig. 2.8(a)], is to be resolved into two components acting along the lines *aa* and *bb*.

   Figure 2.8(a) shows the parallelogram of forces constructed in the usual manner, while Fig. 2.8(b) shows the triangle of forces obtained by the geometric addition of their free vectors.

   The magnitude of the resultant R, angle β and angle γ being known, we may determine the magnitudes of forces P and Q by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha, \qquad \sin\gamma = \frac{P}{R}\sin\alpha$$

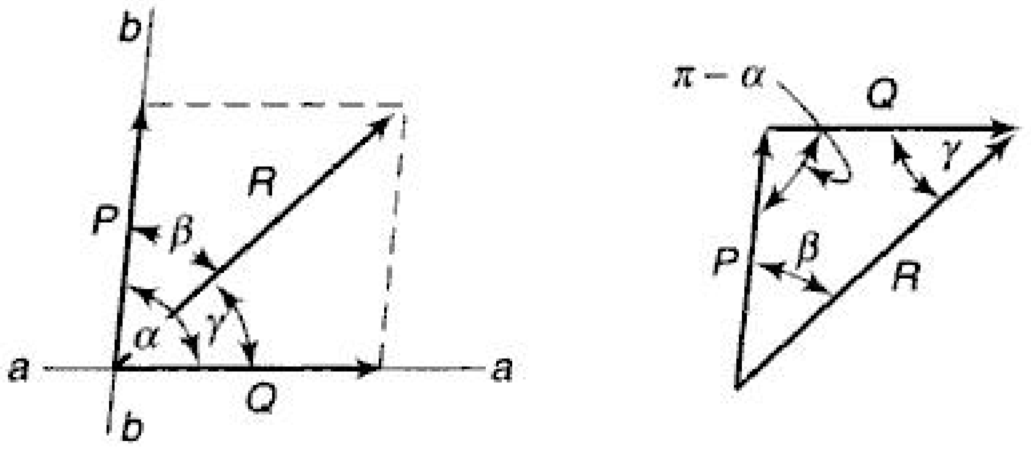


Fig. 2.8 (a) (b)

2. Both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined.

The magnitude and direction of one force can be determined using the above equation.

Equilibrium of Collinear Forces From the principle of the parallelogram of forces, it follows that two forces applied at one point can always be replaced by their resultant which is equivalent to them. Thus, we conclude that two concurrent forces can be in equilibrium, only if their resultant is zero. From the discussion of the previous paragraph it follows that this will be the case if we have two forces of equal magnitude acting in opposite directions along the same line. We shall now generalize this conclusion as the second principle of statics.

Equilibrium Law: *Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction, and collinear in action.*

In engineering problems of statics we often have to deal with the equilibrium of a body in the form of a prismatic bar on the ends of which two forces are acting, as shown in Fig. 2.9. Neglecting their own weights, it follows from the principle just stated that either bar can be in equilibrium only when the forces are equal in magnitude, opposite in direction, and collinear in action, which means that they must act along the line joining their points of application. If these points of application can be assumed to be on the central axis of the bar (as is justifiable in many practical cases), the forces must act along this axis. When such central forces are directed as shown in Fig. 2.9(a), we say that the bar is in *tension*. When they act as shown in Fig. 2.9(b), the bar is said to be in *compression*.

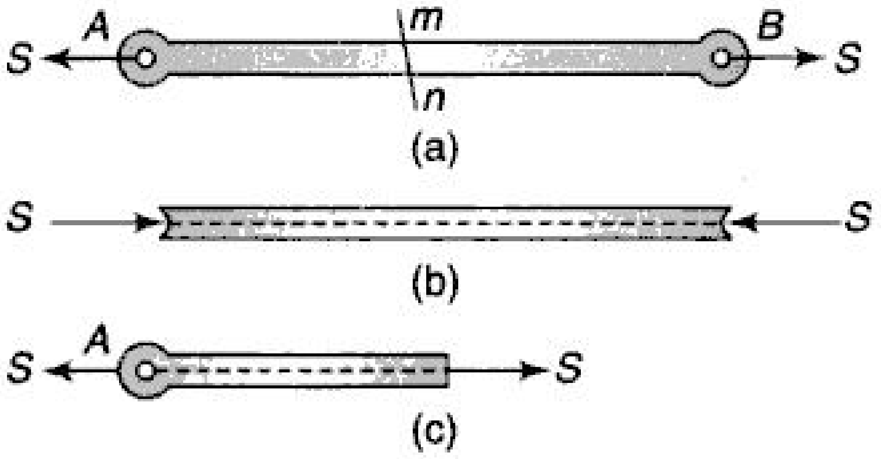


Fig. 2.9

Considering the equilibrium of a portion of the bar *AB* in Fig. 2.9(a), to the left of a section *mn*, we conclude that to balance the external force *S* at *A* the portion to the right must exert on the portion to the left an equal, opposite, and collinear force *S*, as shown in Fig. 2.9(c). The magnitude of this internal axial force which one part of a bar in tension exerts on another part is called the *tensile force in the bar* or simply the *force in the bar,* since in general it may be either a tensile force or a compressive force. Such internal force is actually distributed over the cross-sectional area of the bar, and its *intensity,* i.e., the force per unit of cross-sectional area is called the *stress* in the bar.

Internal forces are the forces which hold together the particles of a body. For example, if we try to pull a body by applying two equal, opposite and collinear forces, an internal force comes into play to hold the body together. Internal forces always occur in pairs and are equal in magnitude, opposite in direction and collinear. Therefore, the resultant of all of these internal forces is zero and does not affect the external motion of the body or its state of equilibrium.

External forces or applied forces are the forces that act on the body due to contact with other bodies or attraction forces from other, separated bodies. These forces may be surface forces (contact forces) or body forces (such as gravitational attraction).

Sometimes we have to deal with the equilibrium of a prismatic bar on each end of which two forces are acting as shown in Fig. 2.10(a), instead of a single force at each end as shown in Fig. 2.9(a) and discussed before. Then the forces at *A* and *B* are replaced with their respective resultants R_a and R_b as shown in Fig. 2.10(b). Now it is the same case as discussed before.

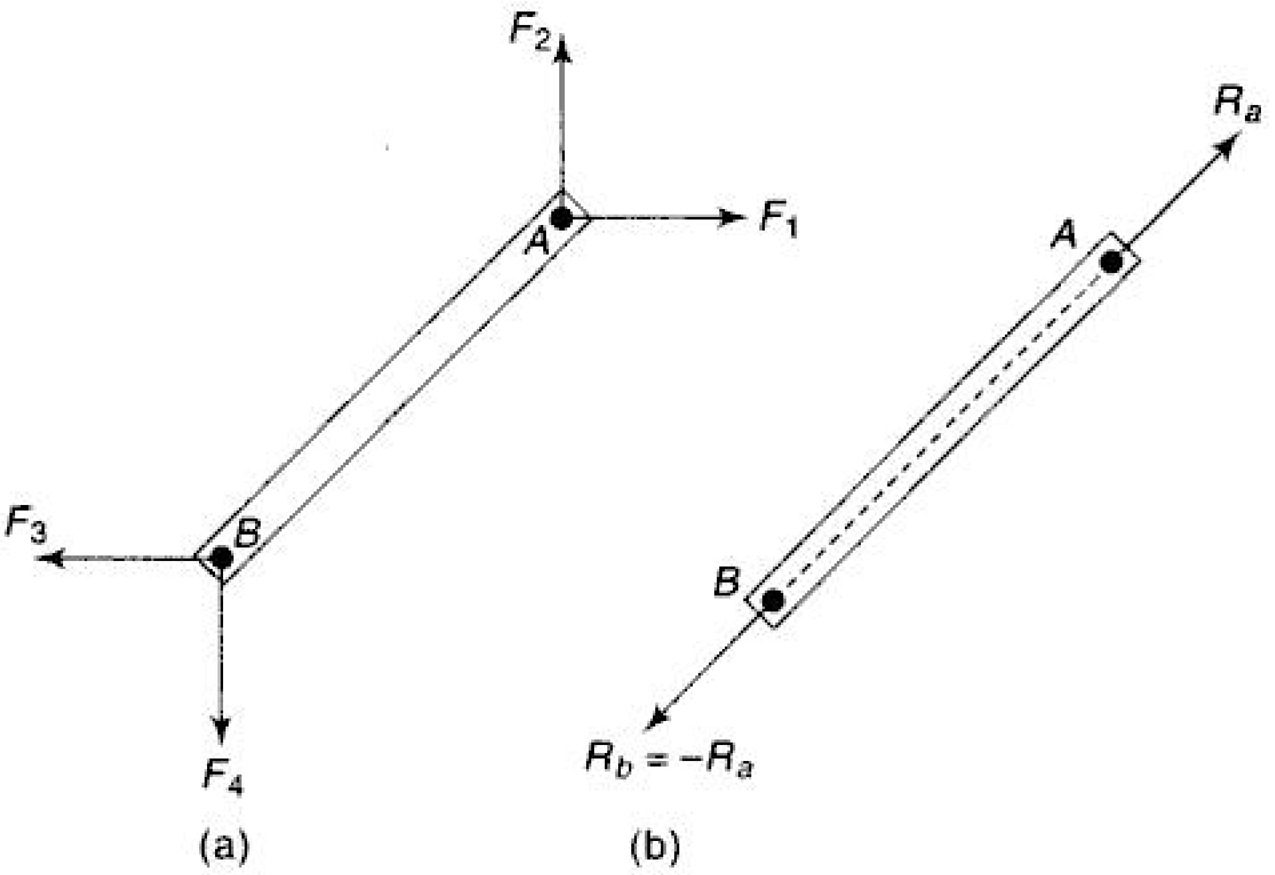


Fig. 2.10 (a) (b)

Other examples of two force members held in equilibrium are shown in Fig. 2.11(a)–(c).

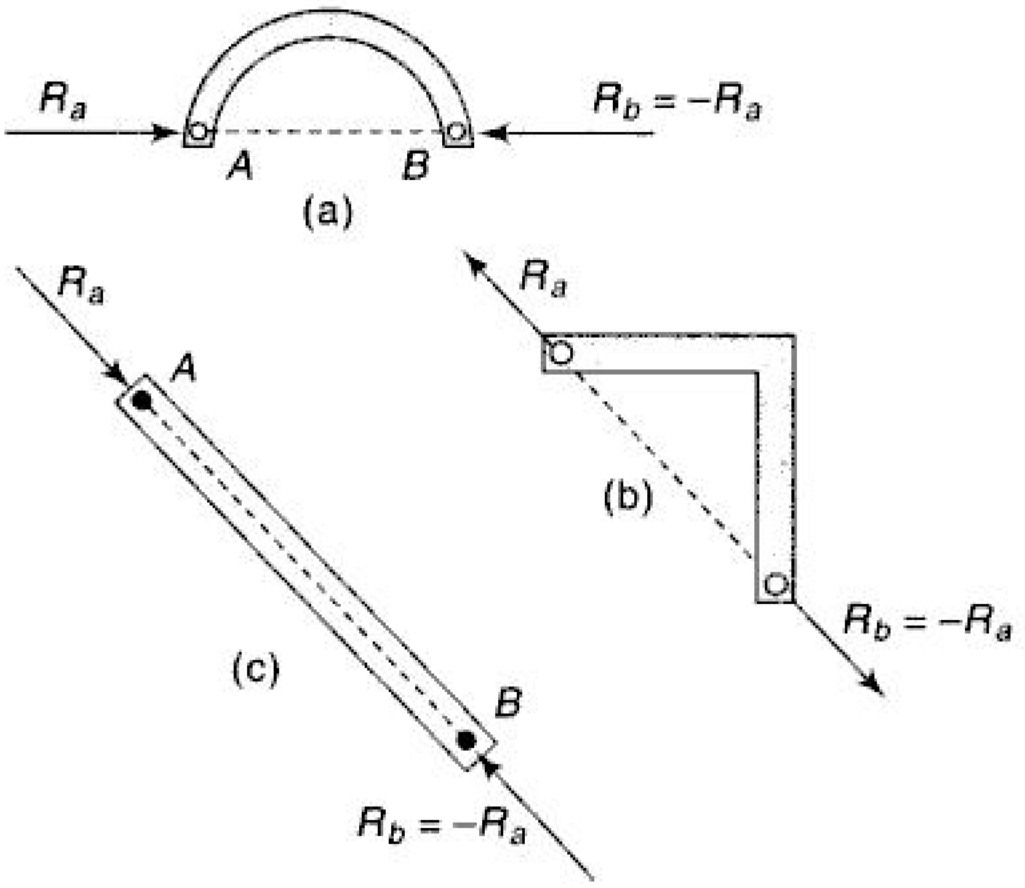


Fig. 2.11

We return now to the case of two forces under an angle α [Fig. 2.4(a)]. From the equilibrium law, we conclude that we can hold these two forces in equilibrium by applying, at point A, a force equal and opposite to their resultant. This force is called the *equilibrant* of the two given forces.

A force, which is equal, opposite and collinear to the resultant of the two given forces is known as the equilibrant of the given two forces.

Superposition and Transmissibility When two forces are in equilibrium (equal, opposite and collinear), their resultant is zero and their combined action on a rigid body is equivalent to that of no force at all. A generalization of this observation gives us the third principle of statics, sometimes called the *law of superposition.*

Law of Superposition: *The action of a given system of forces on a rigid body will in no way be changed if we add to or subtract from them another system of forces in equilibrium.*

Let us consider now a rigid body AB under the action of a force P applied at A and acting along BA as shown in Fig. 2.12(a). From the principle of superposition stated above, we conclude that the application at point B of two oppositely directed forces, each equal to and collinear with P, will in no way alter the action of the given force P. That is, the action on the body of the three forces in Fig. 2.12(b) is identical with the action of the single force P in Fig.2.12(a).

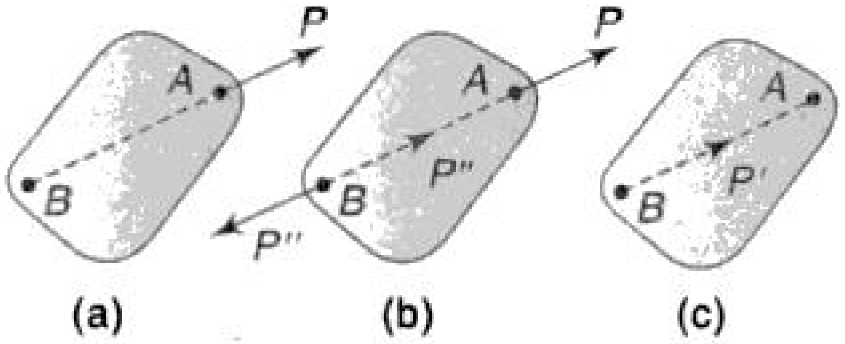


Fig. 2.12

Repeating the same reasoning again, we remove, from the system in Fig. 2.12(b), the equal, opposite, and collinear forces P and P'' as a system in equilibrium. Thus we obtain the condition shown in Fig. 2.12(c) where, instead of the original force P applied at A, we have the equal force P' applied at B. This proves that the point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied. This statement is called the *theorem of transmissibility of a force.*

Theorem of transmissibility of a force: *The point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied.*

By way of an example, let us consider the prismatic bar AB [Fig. 2.13(a)] which is acted upon by two equal and opposite forces P_1 and P_2, applied at the ends and acting along its axis. As discussed before, the bar is in equilibrium under the action of two such forces and is subjected to compression. Now in accordance with the theorem of transmissibility of a force, we transmit P_1 along AB until its point of application is at B and similarly we transmit P_2 along BA to act at A. The condition of the bar now is represented in Fig. 2.13(b), and we see that, while it is still in equilibrium under the action of these forces, the state of compression has been changed to one of tension. Again, imagine that we transmit the point of application of each force to the middle point C of the bar [Fig. 2.13(c)]. The two forces are again in equilibrium, but the bar is now subjected to no internal forces. From this example, we see that, while the transmission of the point of application of a force acting on a body does not change the condition of equilibrium, it may produce a decided change in the internal forces to which the body is subjected. Thus the use of the theorem of transmissibility of a force is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.

The theorem of transmissibility of a force is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.

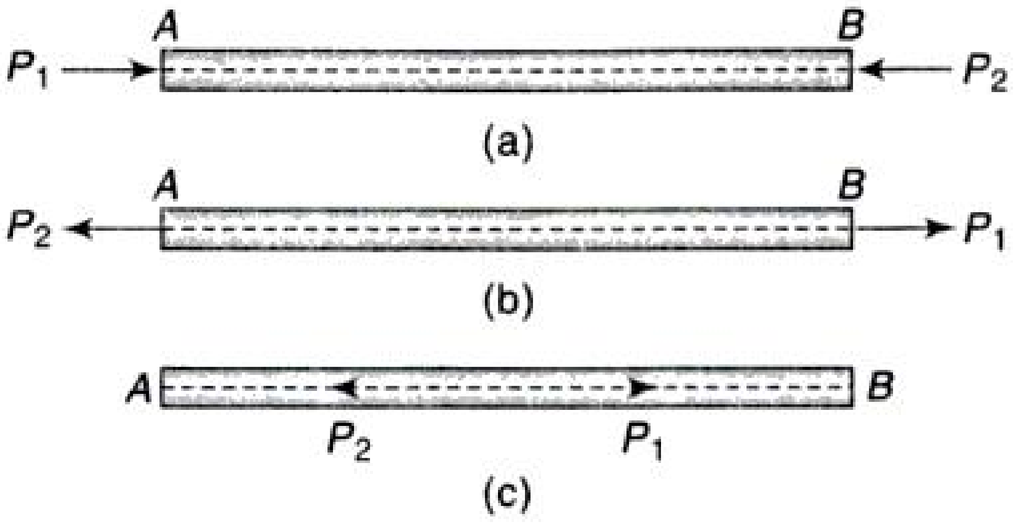


Fig. 2.13

From the theorem of transmissibility of a force it follows that, if two forces *P* and *Q* applied to a body at the points *A* and *B* [Fig. 2.14(a)] are acting along lines intersecting at point *C*, we can transmit the points of application of the forces to point *C* and replace them by their resultant [Fig. 2.14(b)].

If the intersection point *C* is outside the boundary of the body [Fig. 2.14(c)], we assume this point to be rigidly attached to the body by the imaginary extension, indicated in the figure by dotted lines, and then proceed as before.

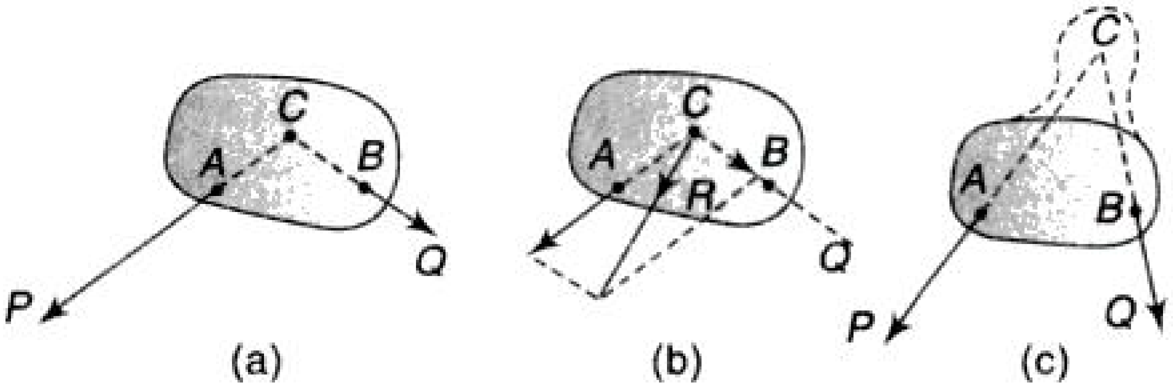


Fig. 2.14

Action and Reaction Very often we have to investigate the conditions of equilibrium of bodies that are not entirely free to move. Restriction to the free motion of a body in any direction is called *constraint.* In Fig. 2.15(a), for example, we have a ball resting on a horizontal plane such that it is free to move along the plane but cannot move vertically downward. Similarly, the ball in Fig. 2.2(a), although it can swing as a pendulum, is constrained against moving vertically downward by the string *AB*. In Fig. 2.16(a), we have a ball of weight *W* supported by a string *BC* and resting against a smooth vertical wall at *A*. With such constraints, all motion of the ball in the plane of the figure is prevented.[2] There are

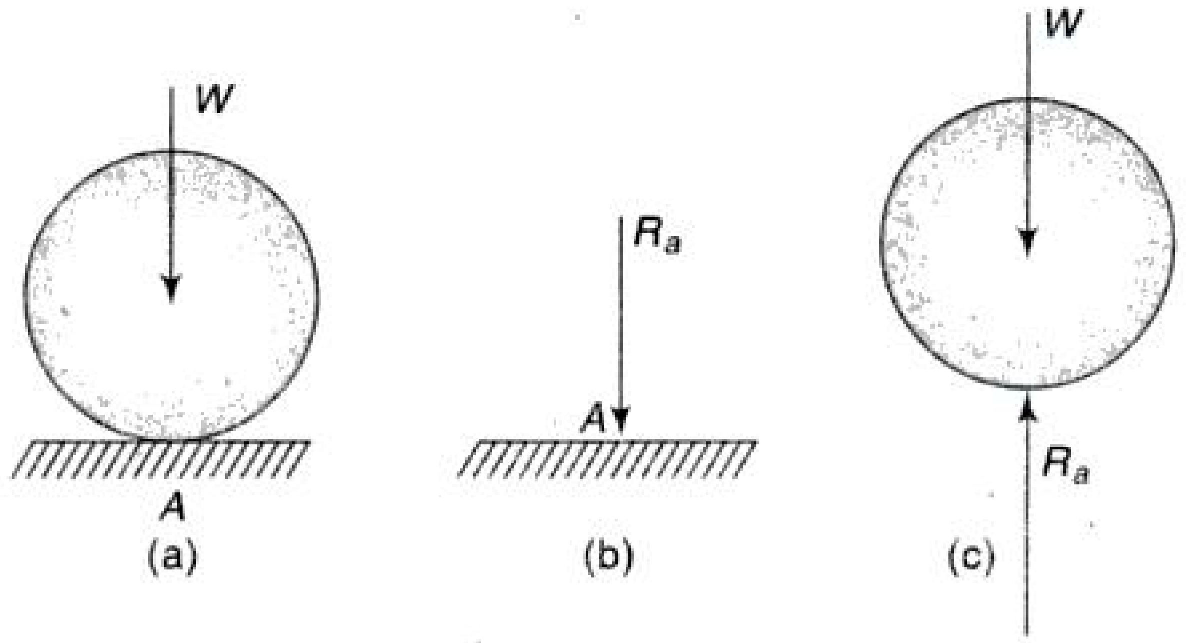


Fig. 2.15

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[2] Since there is no tendency for the ball in this case to move upward or to swing away from the wall, we ignore the fact that the constraints as shown may not be able to prevent such motion.

many other kinds of constraint than those illustrated in Figs. 2.2(a), 2.15(a) and 2.16(a) but these are typical and will suffice as a basis for our present discussion.

A body that is not entirely free to move and is acted upon by some applied force (or forces) will, in general, exert *pressures* against its supports. For example, the ball in Fig. 2.2(a) exerts a downward pull on the end of the supporting string as shown in Fig. 2.2(b). Similarly, the ball in Fig. 2.15(a) exerts a vertical push against the surface of the supporting plane at the point of contact A as shown in Fig. 2.15(b). For the case in Fig. 2.16(a), the ball not only pulls downward on the string BC but also pushes to the left against the wall at A as shown in Fig. 2.16(b). Now in every case, these actions of a constrained body against its supports induce reactions from the supports on the body, and as the fourth principle of statics we take the following statement:

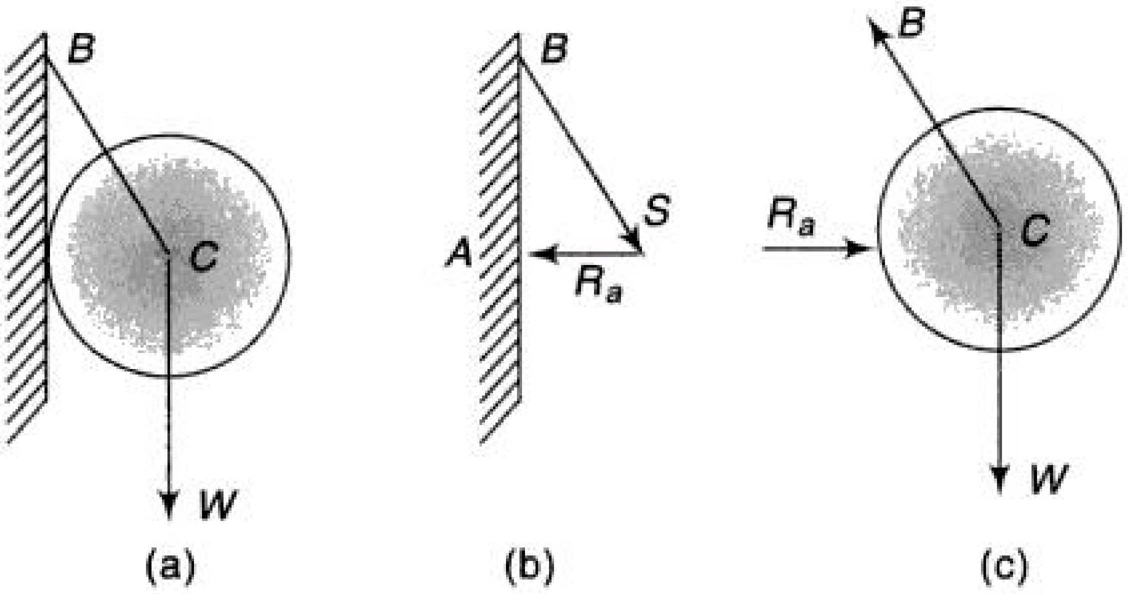


Fig. 2.16 (a) (b) (c)

Law of Action and Reaction *Any pressure on a support causes an equal and opposite pressure from the support so that action and reaction are two equal and opposite forces.* This last principle of statics is of course nothing more than Newton's third law of motion stated in a form suitable for the discussion of problems of statics.

Note ➛ The reaction of a constraint points away from the direction in which the given constraint prevents a body's displacement.

A free body is a body not connected with other bodies and which from any given position can be displaced in any direction in space.

Free-body Diagrams To investigate the equilibrium of a constrained body, we shall always imagine that we remove the supports and replace them by the *reactions* which they exert on the body. Thus, in the case of the ball in Fig. 2.2(a), we remove the supporting string and replace it by the reaction R_a that it exerts on the ball. We know that the point of application of this force must be the point of contact B, and from the law of equilibrium of two forces, we conclude that it must be along the string, i.e., vertical and equal to the weight W; thus it is completely determined. The sketch in Fig. 2.2(c) in which the ball is completely isolated from its support and in which all forces acting on it are shown by vectors is called a free-body diagram.

Free body diagram is a sketch of the isolated body, which shows the external forces on the body and the reactions exerted on it by the removed elements.

The general procedure for constructing a free-body diagram is as follows:

1. A sketch of the body is drawn, by removing the supporting surfaces.

2. Indicate on this sketch all the applied or active forces, which tend to set the body in motion, such as those caused by weight of the body or applied forces; etc.
3. Also indicate on this sketch all the reactive forces, such as those caused by the constraints or supports that tend to prevent motion. (The sense of unknown reaction should be assumed. The correct sense will be determined by the solution of the problem. A positive result indicates that the assumed sense is correct. A negative result indicates that the correct sense is opposite to the assumed sense.)
4. All relevant dimensions and angles, reference axes are shown on the sketch.

Similarly, in the case of the ball in Fig. 2.15(a), we remove the supporting surface and replace it by the reaction R_a that it exerts on the ball. We know that the point of application of this force must be the point of contact A, and from the law of equilibrium of two forces, we conclude that it must be vertical and equal to the weight W; thus it is completely determined. The free-body diagram of the ball in Fig. 2.15(a) is shown in Fig. 2.15(c).

In the case of the ball in Fig. 2.16(a), we again remove the supports and isolate the ball as a free body [Fig. 2.16(c)]. Then besides the weight W acting at C, we have two reactive forces to apply, one replacing the string BC and another replacing the wall AB. Since the string is attached to the ball at C and since a string can pull only along its length, we have the reactive force S applied at C and parallel to BC. Its magnitude remains unknown. Regarding the reaction R_a, we have for its point of application the point of contact A. Furthermore, we assume that the surface of the wall is perfectly smooth so that it can withstand only a normal pressure from the ball. Then, accordingly, the reaction R_a will be horizontal and its line of action will pass through C as shown. Again only the magnitude remains unknown and the free-body diagram is completed. The question of finding the magnitudes of S and R_a will not be discussed here, although it is only necessary to so proportion these vectors that their resultant is equal and opposite to the vertical gravity force W.

From the above discussion, we come across two types of supports namely string support and a smooth surface or support. A flexible weightless and in-extensible string is a constraint prevents a body moving away, from the point of suspension of the string, in the direction of the string. The reaction of the string is directed along the string towards the point of suspension. So, string or cable can support only a tension and this force always acts in the direction of the string. The tension force developed in a continuous string, which passes over a frictionless pulley, must have a constant magnitude to keep the string in equilibrium (Fig. 2.17). Hence, the string or cord, for any angle θ, is subjected to a constant tension S throughout its length.

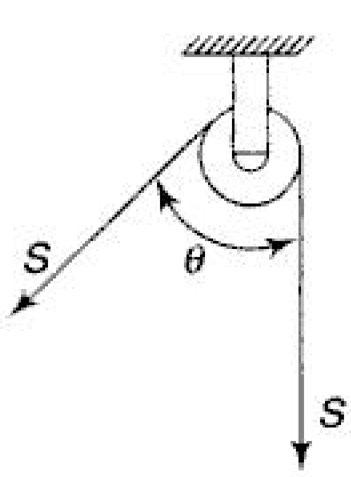


Fig. 2.17

A smooth surface is one whose friction can be neglected. Smooth surface prevents the displacement of a body normal to both contacting surfaces at their point of contact. The reaction of a smooth surface or support is directed normal to both contacting surfaces at their point of contact and is applied at that point

(Fig. 2.18). If one of the contacting surfaces is a point, then the reaction is directed perpendicular or normal to the other surface (Fig. 2.15). If two of the contacting surfaces are points, then the reaction is directed perpendicular or normal to the tangent of contacting surfaces [Fig. 2.19(a) and 2.20(a)].

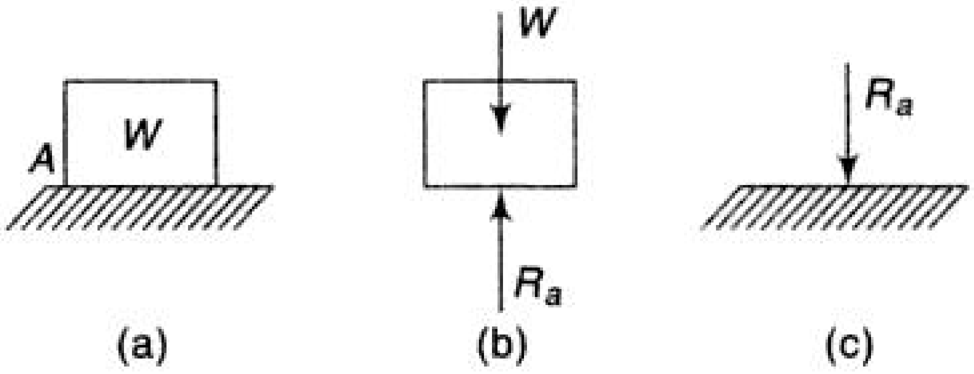


Fig. 2.18

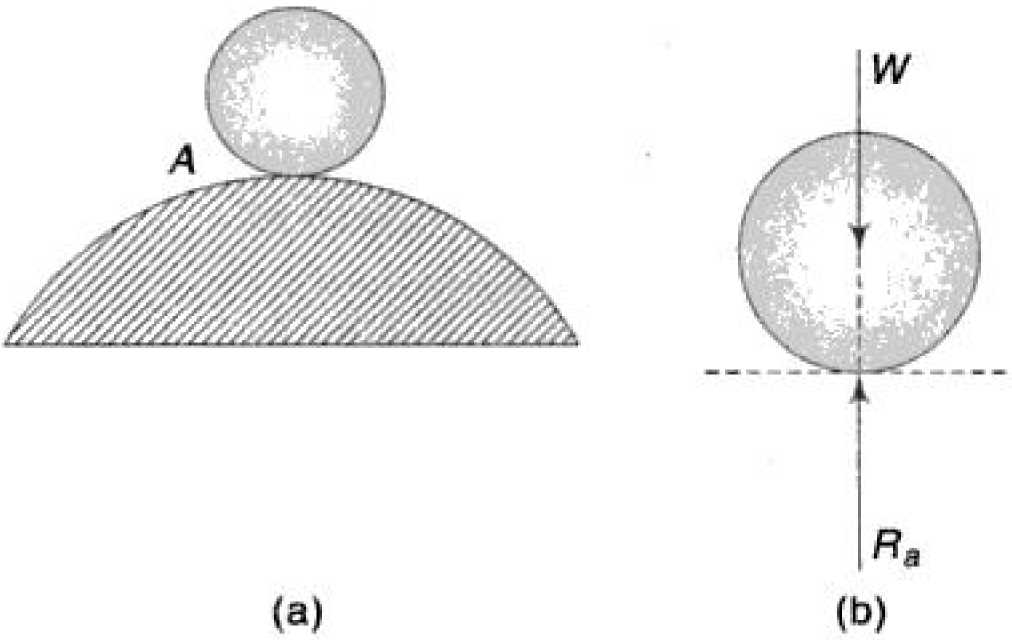


Fig. 2.19

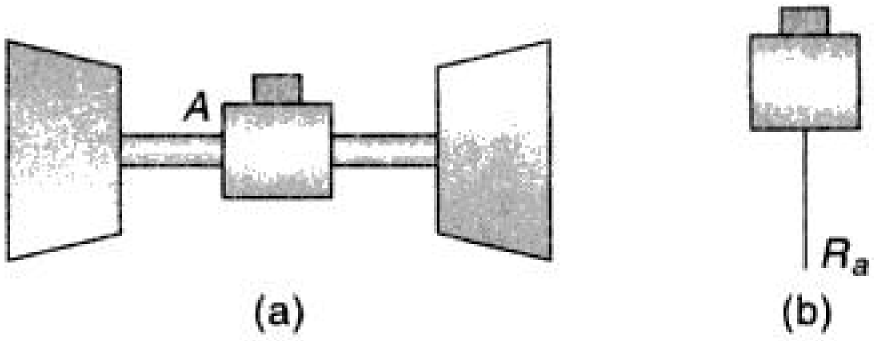


Fig. 2.20

The free body diagrams of the bodies are shown in Figs 2.15(b), 2.18(b), 2.19(b), 2.20(b), respectively.

Another type of support is linear elastic spring (Fig. 2.21). The magnitude of force developed by a linear elastic spring which has a stiffness k, and is deformed a distance x measured from its unloaded position, is

$$S = kx$$

Note ➤➤ x is determined from the difference in the spring's deformed length and its initial length. If x is positive, S 'pulls' on the spring; whereas if x is negative, S must 'push' on it.

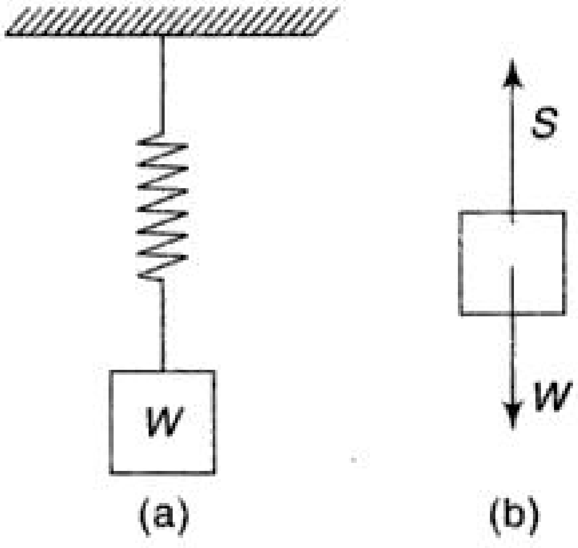


Fig. 2.21

In the case of the body in Fig. 2.21(a), we remove the supporting spring and replace it by the spring force S that it exerts on the body. We know that the point of application of this force must be the point of contact, and from the law of equilibrium of two forces, we conclude that it must be along the spring, i.e., vertical and equal to the weight W;

thus it is completely determined. The free body diagram of the body in Fig. 2.21(a) is shown in Fig. 2.21(b).

One more example of free-body diagram is considered here. The lawn roller, of weight W, being pushed up the inclined smooth plane as shown in Fig. 2.22(a). In the case of the lawn roller in Fig. 2.22(a), we again remove the support and isolate the body as a free body [Fig. 2.22(b)]. Then beside the weight *W* and push *P* acting at centre *O*, we have one reactive force R_a to apply, replacing the inclined plane. The reactive force R_a have its point of application at the point of contact *A*. We assume that the surface of the inclined plane is perfectly smooth so that it can withstand only a normal pressure from the roller. Then the reaction R_a will be normal to the inclined surface and its line of action will pass through *O* as shown here the magnitude remains unknown and the free body diagram is completed the question of finding the magnitudes of *P* and R_a will not be discussed here.

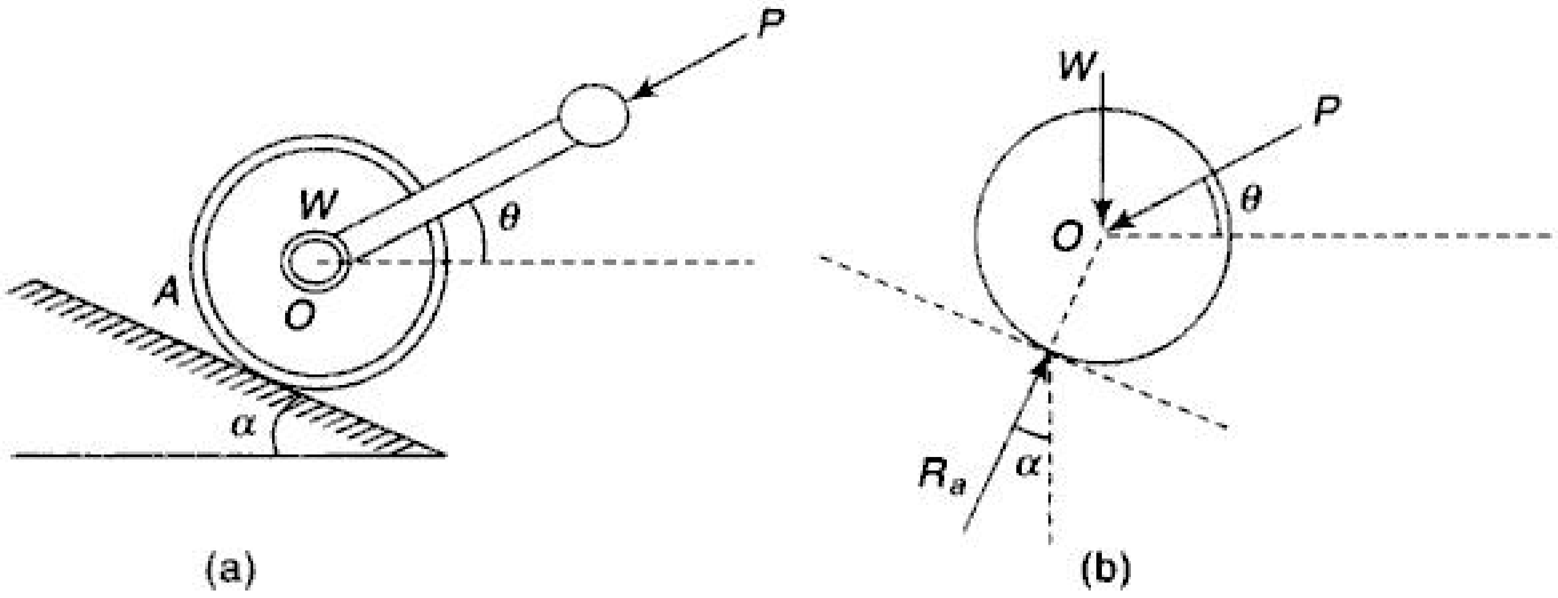


Fig. 2.22

Proceeding as above with constrained bodies, we shall always obtain two kinds of forces acting on the body: the given forces, usually called *active forces,* such as the gravity force *W* in Fig. 2.16(c), and *reactive forces,* replacing the supports, such as the forces *S* and R_a in Fig. 2.16(c). To have equilibrium of the body, it is necessary that the active forces and reactive forces together represent a system of forces in equilibrium. Thus it is by means of the free-body diagram that we define the system of forces with which we must deal in our investigation of the conditions of equilibrium of any constrained body. The construction of this diagram should be the first step in the analysis of every problem of statics, and it must be evident that any errors or omissions here will reflect themselves on all subsequent work.

The essential problem of statics may now be briefly recapitulated as follows: We have a body either partially or completely constrained which remains at rest under the action of applied forces. We isolate the body from its supports and show all forces acting on it by vectors, both active and reactive. We then consider what conditions this system of forces must satisfy in order to be in equilibrium, i.e., in order that they will have no resultant.

Examples Examples Examples Examples Examples

1. Two very nearly parallel forces P and Q are applied to a rigid body at points A and B, as shown in Fig. 2.23. Find their resultant R graphically.

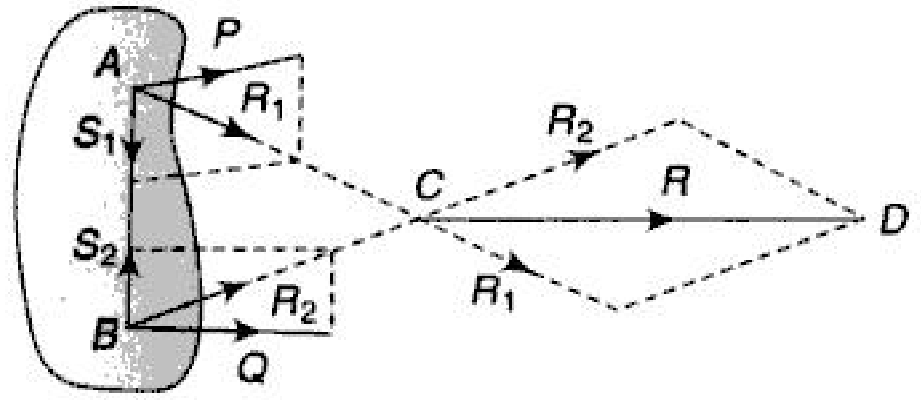
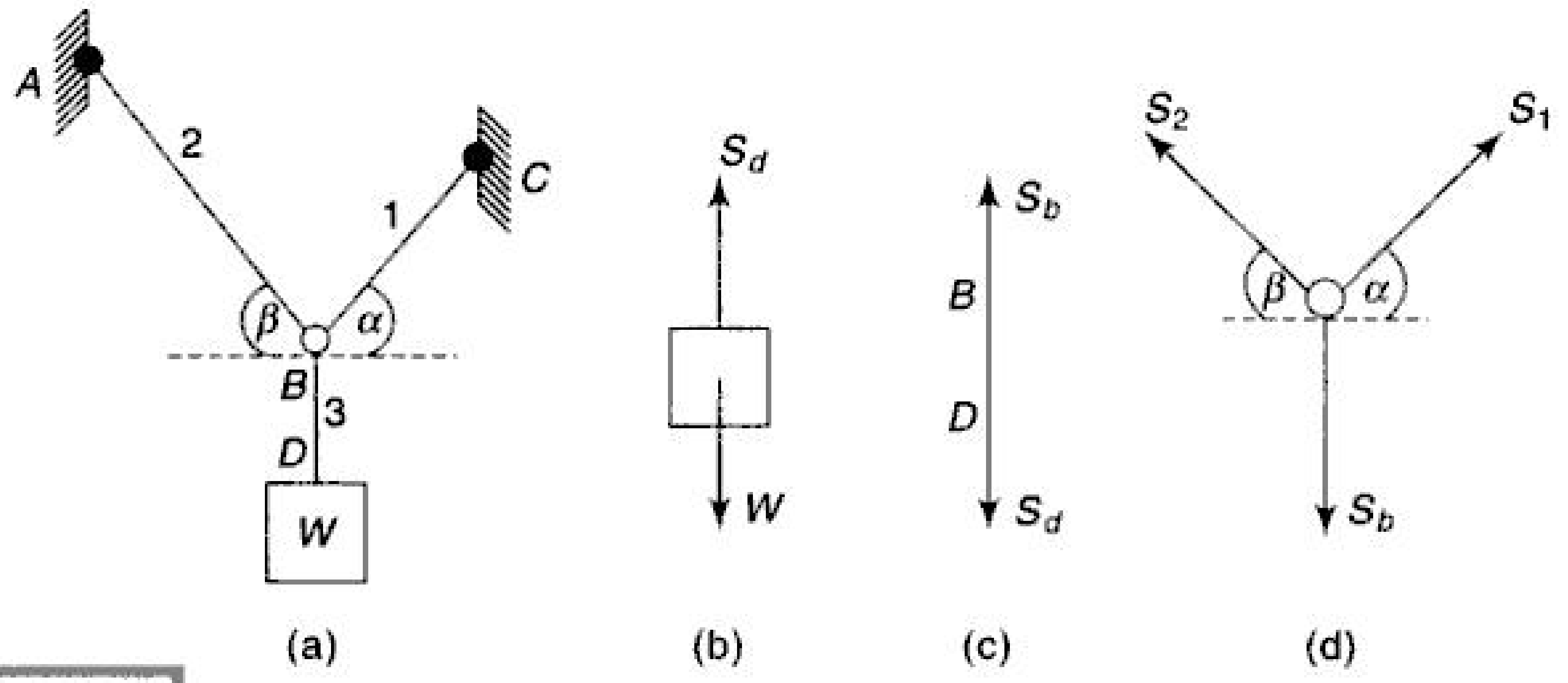


Fig. 2.23

Solution: Since the point of intersection of the given forces P and Q is not well defined and does not occur within the limits of the drawing, we begin by adding to the system two equal, opposite and collinear forces S_1 and S_2 of any convenient magnitudes at points A and B, as shown in the figure. It follows from the law of superposition that two such forces, being in equilibrium, do not change the action of the given forces P and Q. Hence the resultant R_1 of P and S_1 together with the resultant R_2 of Q and S_2, obtained as shown, are statically equivalent to the given forces P and Q and their resultant R will be the one required. To find this resultant R, we transmit R_1 and R_2 along their lines of action to point C, which is a well-defined point, and complete the parallelogram of forces as shown. The vector $\overline{CD}$ represents the required resultant and if all constructions have been made to scale, its magnitude may be measured directly from the drawing.

2. Draw a free body diagram of the body of weight W, the string BD and the ring shown in Fig. 2.24(a).



(a) (b) (c) (d)

Fig. 2.24

Solution: In the case of the body in Fig. 2.24(a), remove the support, i.e., string and isolate the body as a free body [Fig. 2.24(b)]. Replace the string support by the tension S_3 or S_d it exerts on the body. We know that the point of application of this force must be the point of contact D, and from the law of equilibrium of two forces, we conclude that it must be along the string, i.e., vertical and equal to the weight W; thus it is completely determined.

If the string BD isolated from its supports, then there are only two forces acting on it, Fig. 2.24(c), namely the tensile force of the string S_d and the tensile force S_b caused by the string. S_d shown here is equal but opposite to that shown in Fig. 2.24(b) from the law of action and reaction.

From the law of equilibrium of two forces S_d and S_b are equal, i.e., $S_d = S_b = S_3 = W$.

Now isolate the ring at B from its supports. On the ring, three forces are acting. All the forces on the ring are reactive forces from the strings and shown as in Fig. 2.24(d).

Note ↠ S_b shown here is equal but opposite to that shown in Fig. 2.24(c) from the law of action and reaction.

3. The body of weight W is supported as shown in Fig. 2.25(a). Draw a free body diagram of the body and the knot at C.

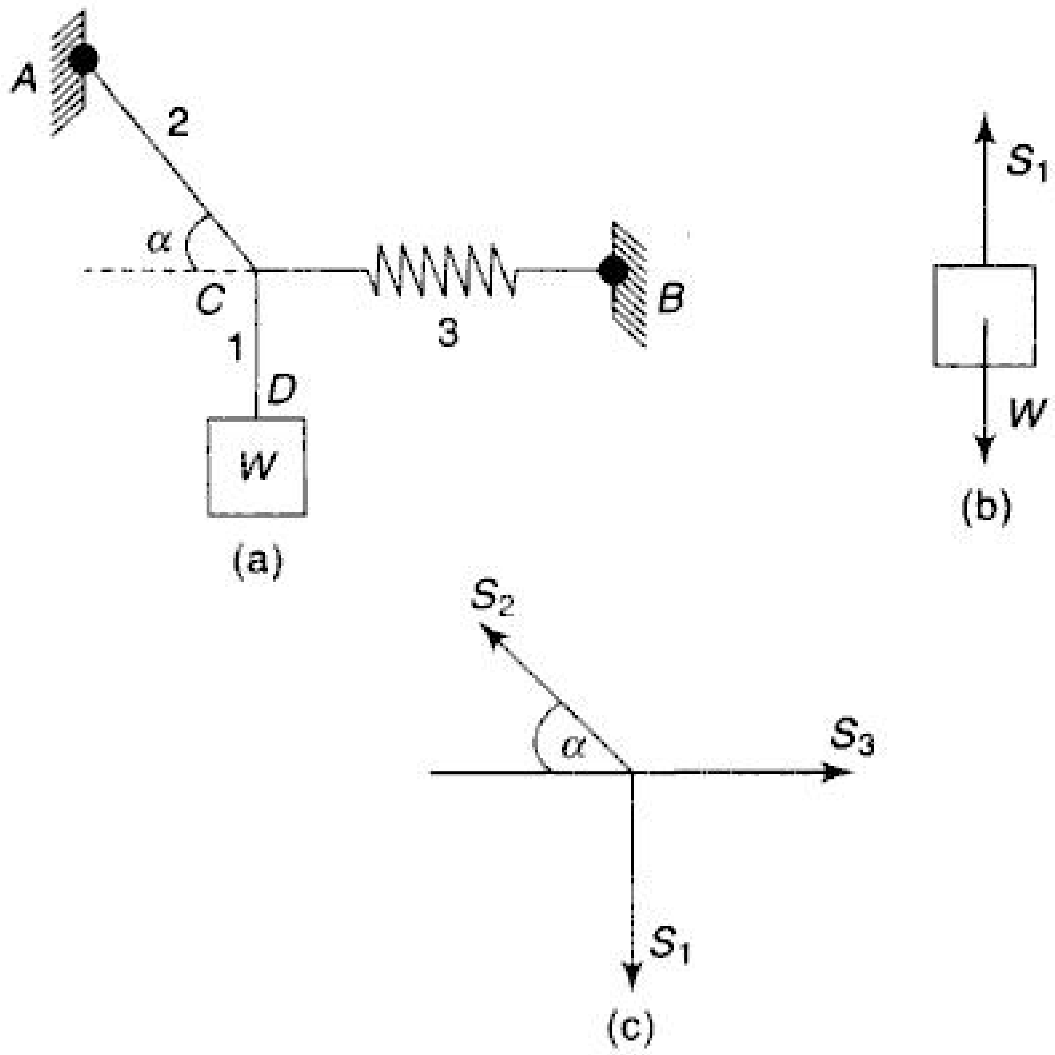


Fig. 2.25

Solution: Isolate the body from its supports. There are two forces acting on the body, namely, its weight and the tensile force S_1 of the string CD. The free body diagram of the body shown in Fig. 2.25(b). From the law of equilibrium of two forces, we have $S_1 = W$ and S_1 is acting vertically upward.

If we isolate the knot from its supports, there are three forces acting on it. They are reactive forces from the spring and the tensile forces in the string. The free body diagram of the knot is shown in Fig. 2.25(c) as usual.

4. A body of weight W is supported on a frictionless pulley as shown in Fig. 2.26(a). Draw a free body diagram of the body and the pulley, if the radius of the pulley and the weight of the pulley are neglected.

Solution: If we isolate the body from its supports, there are two forces acting on the body. One is weight and the other is tension of the string. The free body diagram is as shown in Fig. 2.26(b).

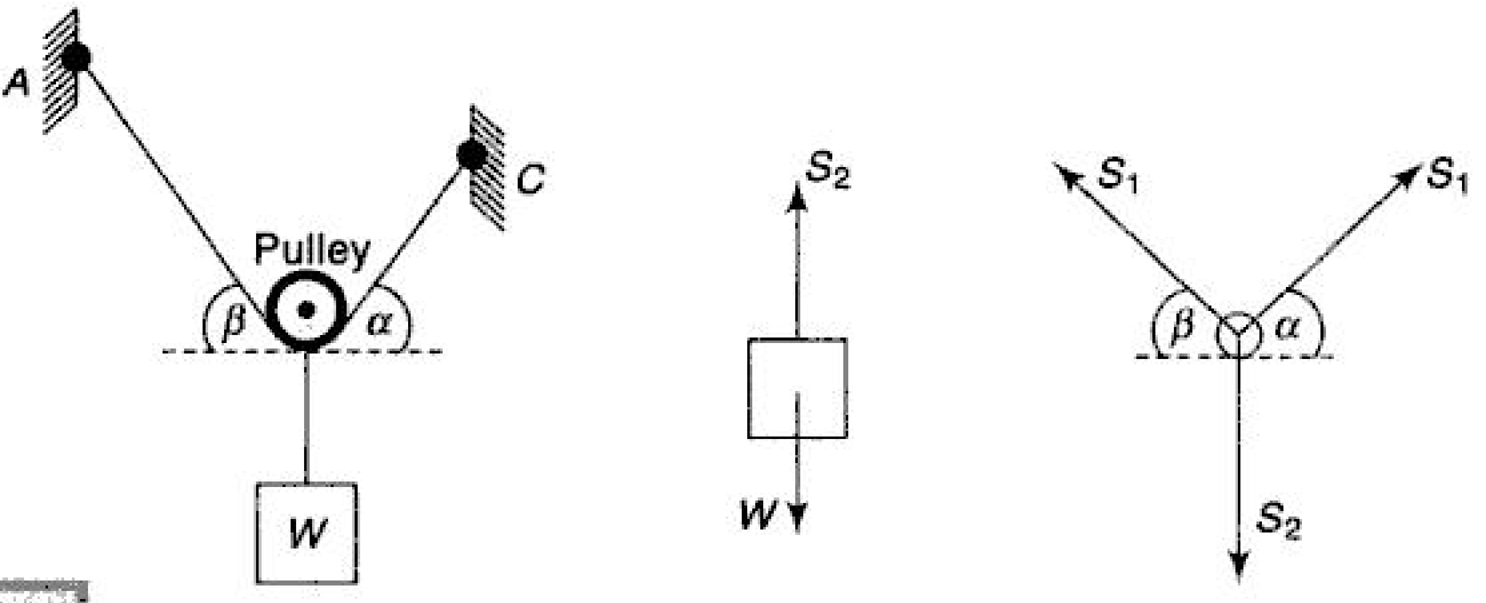


Fig. 2.26 (a) (b) (c)

From the law of equilibrium of two forces, we obtain $S_2 = W$ and S_2 is acting vertically upward. Now isolate the pulley from its supports. As the pulley is frictionless, the forces exerted at B by the portions AB and BC of the cord must be equal. These two forces can balance the vertical resultant, this condition requires that AB and BC be equally inclined to the horizontal, i.e., $\beta = \alpha$. The free body diagram of the pulley is as shown in Fig. 2.26(c).

5. Two spheres of weight P and Q rest inside a hollow cylinder, which is resting on a horizontal plane as shown in Fig. 2.27. Draw the free body diagram/s of both the spheres taken together and both the spheres taken separately.

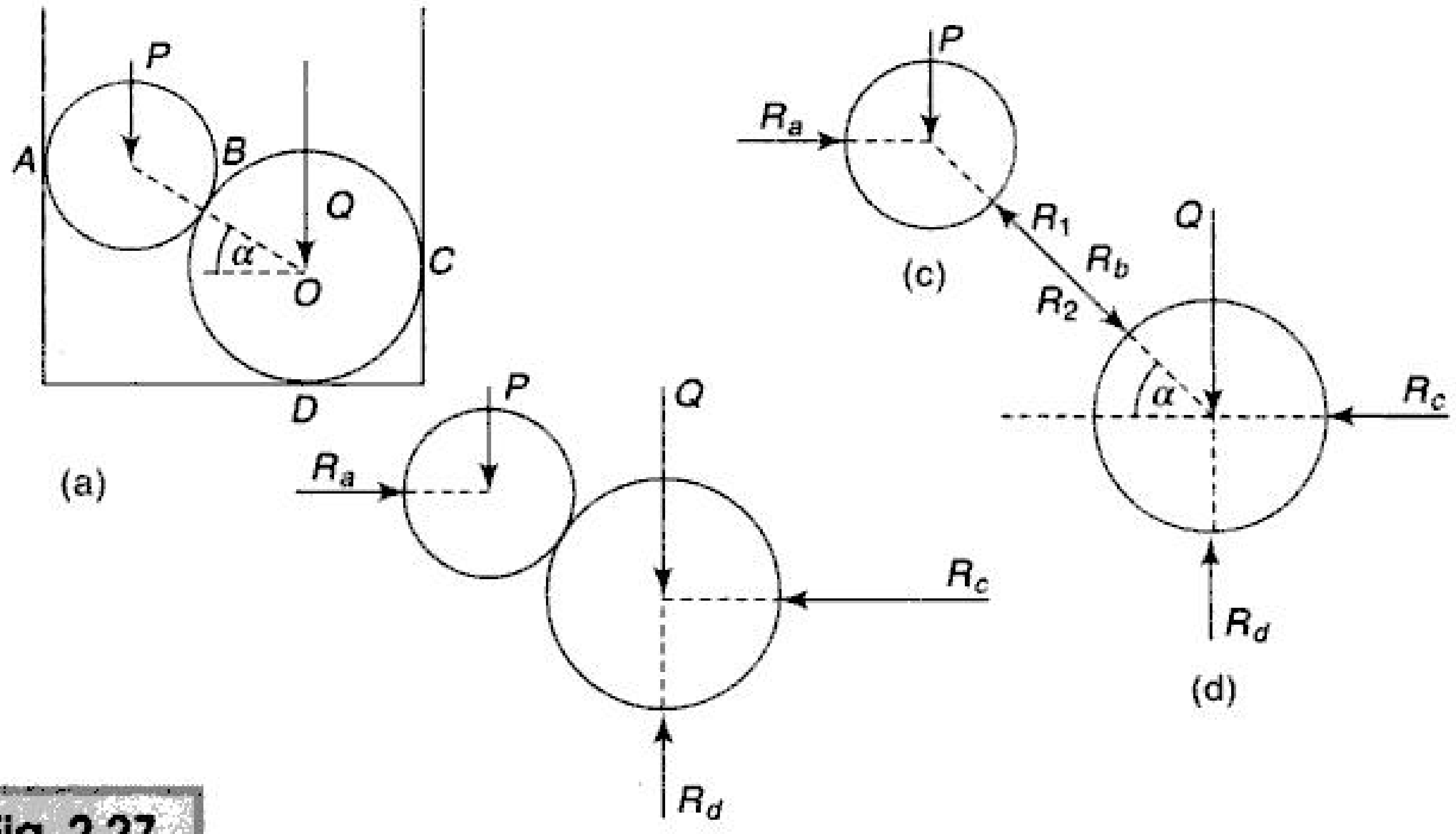


Fig. 2.27 (b)

Solution: Since the spheres are smooth, the pressures at the various points of contact must be normal to the surfaces. Removing the supporting walls and floor and replacing them by their reactions R_a, R_c and R_d, we obtain the free body diagram for both spheres as shown in Fig. 2.27(b). These reactions are equal and opposite to the required pressures exerted by the spheres on the walls and floor.

Isolate the spheres taken separately from its supports. On the sphere P, three forces are acting, namely, weight, the reaction from the wall of the cylinder. The free body diagram is as shown in Fig. 2.27(c). On the sphere Q, four forces are acting, namely, weight, the reaction from the wall of the cylinder, reaction from the floor and the reaction from the sphere P. The free body diagram is as shown in Fig. 2.27(d).

At the point of contact between the two spheres, we have two equal and opposite forces R_a, R_2 which must act along the line OE joining the centres of the spheres. When considering the free body diagram of the upper sphere, we take only the force R_1, representing the reaction exerted by the lower sphere, likewise when considering the lower sphere, we take only the force R_2.

6. Find the force with which the 1000 N press against the floor shown in Fig. 2.28(a).

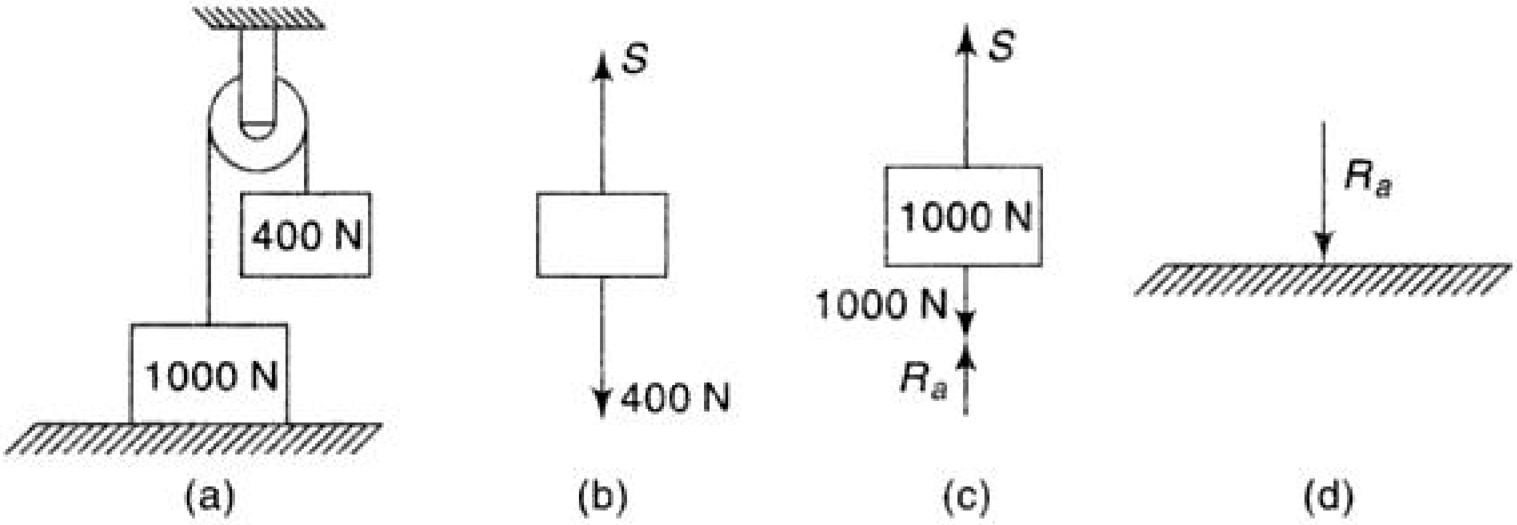


Fig. 2.28

Solution: The free body diagram of 400 N weight is as shown in Fig. 2.28(b).

From the law of equilibrium of two forces, we get

$$S = 400 \text{ N}$$

Removing the supporting string and the floor and replacing them by their reactions S and R_a we obtain the free body diagram of the body as shown in Fig. 2.28(c). The reactions S and R_a are equal and opposite to the required pressures exerted by the body on the string and floor. The free body diagram of the floor is shown in Fig. 2.28(d).

The three forces S, R_a and W are all acting along one line, the sides of the polygon of forces will all lie along one line and the geometric summation will be replaced by algebraic summation. Therefore, the resultant is the algebraic sum of the three forces, i.e.,

$$R = S + R_a - 1000$$

The resultant is equal to zero, since the body is in equilibrium. Equating the resultant zero, we get

$$R_a = 600 \text{ N}$$

Therefore, the force with which the 1000 N press against the floor = 600 N.

7. Determine the components of the 1000 N force shown along the aa' and bb' axes shown in Fig. 2.29(a).

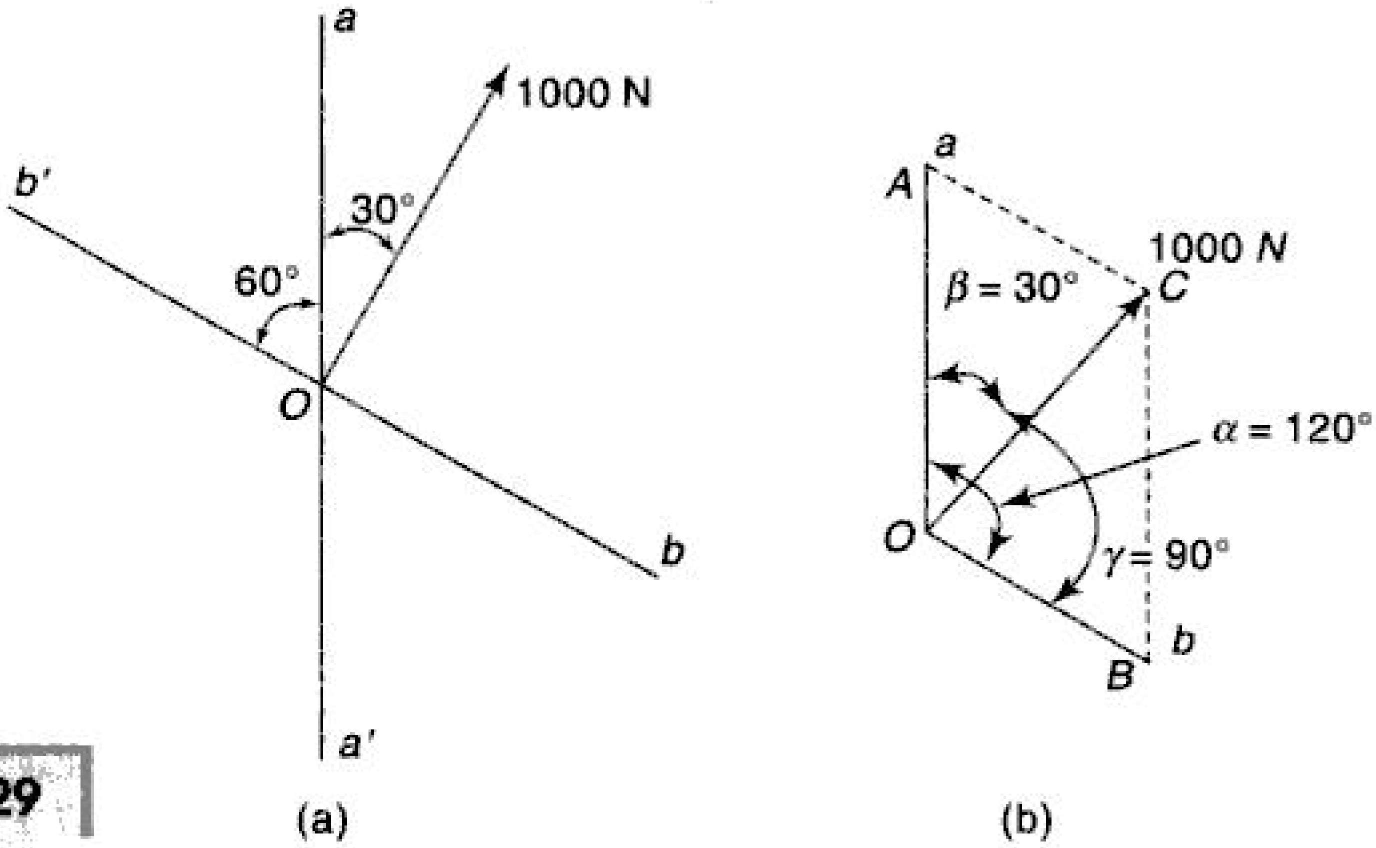

a
1000 N
b'
30°
60°
O
b
a'
a
A
1000 N
β = 30°
C
α = 120°
O
γ = 90°
b
B

Fig. 2.29 (a) (b)

Solution: Figure 2.29(b) shows the parallelogram of forces *OACB* constructed in the usual manner. The angles β and γ are given by the equations

$$\sin \beta = \frac{Q}{R} \sin \alpha, \qquad \sin \gamma = \frac{P}{R} \sin \alpha \tag{c}$$

Substituting the values $\alpha = 120°$, $\beta = 30°$, $\gamma = 90°$ and $R = 1000$ N, we get from Eq. (c)

$$P = 1154.7 \text{ N}$$
$$Q = 577.35 \text{ N}$$

8. The 500 N force is to be resolved into components along the *aa* and *bb* axes as shown in Fig. 2.30(a). Determine the angles β and γ knowing that the component along *aa′* is to be 400 N. Also, find the corresponding value of the component along *bb′* axis.

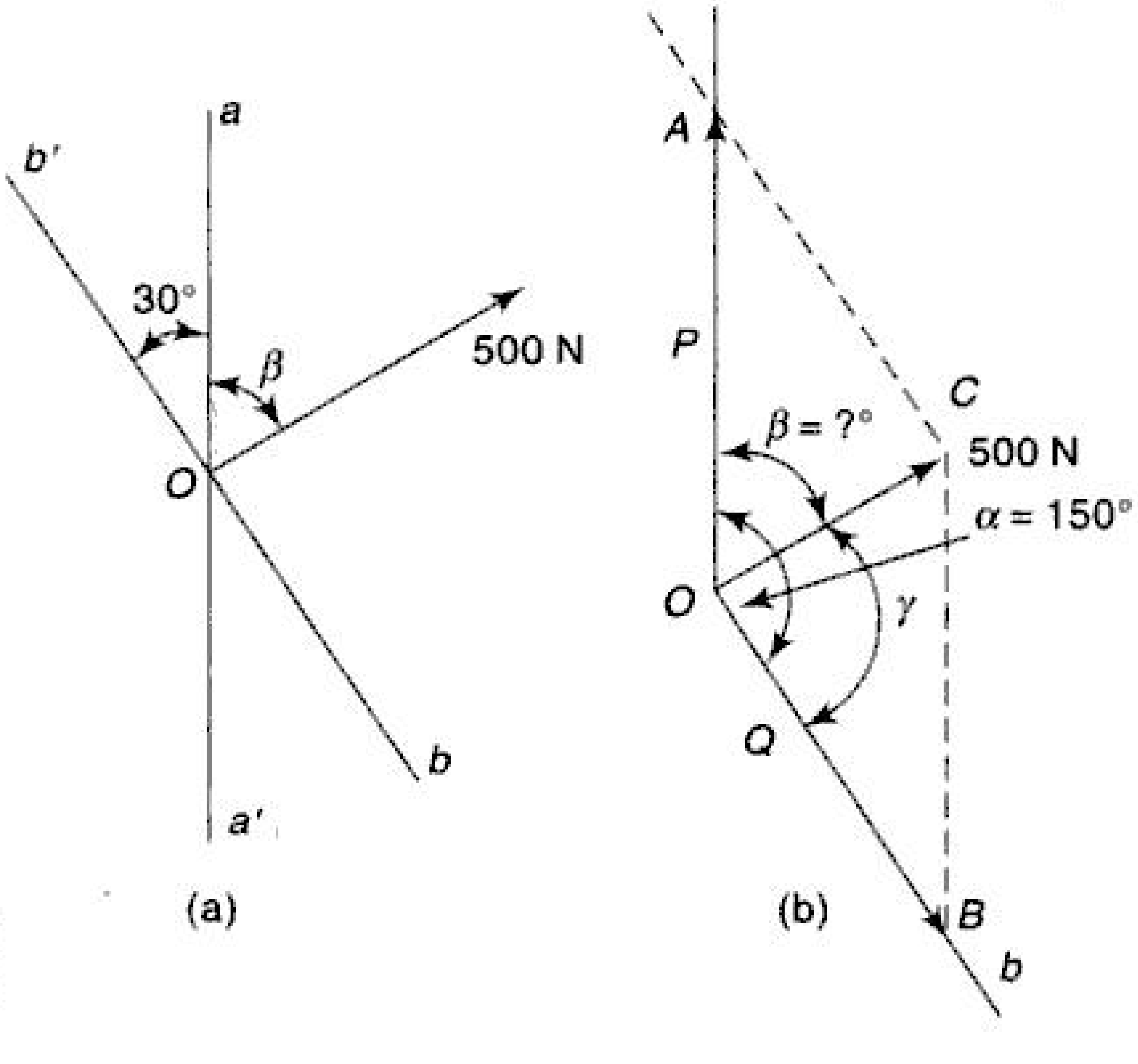

a
b'
30°
β
500 N
O
b
a'
A
P
C
β = ?°
500 N
α = 150°
O
γ
Q
B
b

Fig. 2.30 (a) (b)

Solution: In this problem both the direction and the magnitude of one component are given; the direction and the magnitude of the other, to be determined. Using Fig. 2.30(b) and the relation

$$\sin \gamma = \frac{P}{R} \sin \alpha \tag{d}$$

and substituting the values $\alpha = 150°$, $P = 400$ N and $R = 500$ N, we will get from Eqn. (d)

$$\gamma = 23.58°$$

and $\beta = \alpha - \gamma = 150 - 23.58 = 126.42°$

Using Fig. 2.30(b) and the relation

$$\sin \beta = \frac{Q}{R} \sin \alpha \tag{i}$$

and substituting the values $\alpha = 150°$, $\beta = 126.42°$, and $R = 500$ N, we will get from Eq. (e).

$$Q = 804.7 \text{ N}$$

Important Terms and Concepts

Statics
Fluid mechanics
Concentrated load
Free vectors
Resultant
Rectangular components
Equilibrium law
Stress
Theorem of transmissibility
Law of action and reaction
Free body diagram
Rigid Body
Force
Vector
Composition of forces
Triangle law
String
Tension
Equilibrant
Constraint
Active forces
Strength of materials
The specifications of a force
System of forces
Parallelogram law
Resolution of forces
Smooth surface
Compression
Law of superposition
Action and reaction
Reactive forces

SUMMARY

- Statics deals with the conditions of equilibrium of bodies acted upon by forces.
- A rigid body is defined as a definite quantity of matter, the parts of which are fixed in position relative to one another. The physical bodies are never absolutely rigid but deform slightly under the action of loads, which they have to carry. If the deformation is negligible when compared with the size of the body, it is assumed to be rigid.
- Force may be defined as any action that tends to change the state of rest of a body to which it is applied. The three quantities, which completely define the force, are called its *specifications*. The specifications of a force are (1) its magnitude, (2) its point of application, and (3) its direction. The SI units used by engineers to measure the magnitude of a force are the newton (N). The point of application of a force acting upon a body is that point in the body at which the force can be assumed to be concentrated. The direction of a force is the direction, along a straight line through its point of application, in which the force tends to move a body to which it is applied.
- The point of application at which the total weight can be assumed to be concentrated is called the center of gravity of the body.

- Any quantity, such as force, that possesses direction as well as magnitude is called a *vector quantity* and can be represented graphically by a segment of a straight line.
- Force can be represented graphically by a segment of a straight line. The length of the straight line segment, to some convenient scale, shows the magnitude of the force. The arrow indicates the direction of force. Either the beginning (tail) or the end (head) of a vector may be used to indicate the point of application of the force.
- System of forces: When several forces of various magnitudes and directions act upon a body, they are said to constitute a system of forces.
- Parallelogram law of forces states that if two forces acting simultaneously at a point may be represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represented in magnitude and direction by diagonal of the parallelogram, which passes through their points of intersection.
- Triangle law of forces: If two forces acting simultaneously on a particle represented by the two sides of a triangle (in magnitude and direction) taken in order, then their resultant is represented by the third side (closing side) taken in an opposite order.
- Equilibrium law: Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction, and collinear in action.
- A force, which is acting on a bar, tends to elongate the bar is known as tensile force. The magnitude of the internal axial force which one part of a bar in tension exerts on another part is called the tensile force in the bar.
- A force, which is acting on a bar, tends to shorten the bar is known as compressive force. The magnitude of the internal axial force which one part of a bar in compression exerts on another part is called the compressive force in the bar.
- A force, which is equal, opposite and collinear to the resultant of the two given forces is known as the equilibrant of the given two forces.
- Law of superposition: The action of a given system of forces on a rigid body will in no way be changed if we add to or subtract from them another system of forces in equilibrium.
- The point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied. This statement is called the *theorem of transmissibility of a force. The theorem of transmissibility of a force* is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.
- Restriction to the free motion of a body in any direction is called *constraint.*
- Law of action and reaction: *Any pressure on a support causes an equal and opposite pressure from the support so that action and reaction are two equal and opposite forces.*
- Forces that act on a constrained body can be divided into two kinds of forces: active forces and reactive forces. Forces acting on a body that are given forces such as gravity forces are called active forces. Reactive forces are those forces that are exerted on a body by the supports to which it is attached.
- Smooth surface or support: A smooth surface is one whose friction can be neglected. Smooth surface prevents the displacement of a body normal to both contacting surfaces at their point of contact. The reaction of a smooth surface or support is directed normal to both contacting surfaces at their point of contact and is applied at that point. If one of the contacting surfaces is a point, then the reaction is directed perpendicular or normal to the other surface.
- A constraint provided by a flexible in-extensible string prevents a body moving away from the point of suspension of the string in the direction of the string. The reaction of the string is directed along the string towards the point of suspension.

- Sometimes a linear elastic spring is used as a support. The reactive force of a spring may either be tensile or compressive in nature, according to the load acting on the spring.
- Free body diagram is a sketch of the isolated body, which shows the external forces on the body and the reactions exerted on it by the removed elements.
- The essential problem of statics: Consider a body, either partially or completely constrained, which remains at rest under the action of applied forces. Isolating the body from its supports and showing all forces acting on it by vectors, both active and reactive. Then considering what conditions this system of forces must satisfy in order to be in equilibrium, i.e., in order that they will have no resultant.

Important Formulae

The magnitude of the resultant R of the two forces P and Q having an angle α between them and acting at a point, is given by the equation

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

The angles β and γ may be determined by using the equations

$$\sin\beta = \frac{Q}{R}\sin\alpha \qquad \sin\gamma = \frac{P}{R}\sin\alpha$$

where, β = angle between the forces R and P

γ = angle between the forces R and Q

PRACTICE SET 2.1

Review Questions

1. What do you study in statics?
2. Define rigid body.
3. State and explain the concept of force.
4. Give the examples of force.
5. What is meant by specifications of a force?
6. Explain the term concentrated force.
7. Define vector and free vector.
8. State and explain the parallelogram law of forces.
9. State and explain the triangle law of forces.
10. Explain rectangular components of a force.
11. State the law of equilibrium of two forces.
12. Discuss about two force body or two force members.
13. Define tension and compression in a bar.
14. State the law of superposition of forces.
15. Explain the theorem of transmissibility of a force.
16. What is the limitation of the theorem of transmissibility of a force?
17. State the law of action and reaction.
18. Define constraint and explain the constraints string, smooth surface and linear elastic spring .
19. Differentiate between active and reactive forces.
20. What is a 'free body diagram'? Explain it with the help of figures.
21. State the essential problem of statics.

Objective Questions

1. Force can be characterized by
   (a) point of application (b) magnitude, direction
   (c) direction (d) point of application, magnitude and direction
   [*Ans.* (d)]
2. The forces whose lines of action lie in the same plane and are meeting at one point, are known as
   (a) coplanar concurrent force system
   (b) coplanar non-concurrent force system
   (c) non-coplanar concurrent force system
   (d) non-coplanar non-concurrent force system [*Ans.* (a)]
3. The resultant of two forces can be defined as a force that
   (a) keeps the system in equilibrium
   (b) has the greatest magnitude in the system
   (c) has the same effect as the two forces
   (d) has the same effect as one force [*Ans.* (c)]
4. In the case of gravity force distributed throughout the volume of a body, the point of application at which the total weight can be assumed to be concentrated is called
   (a) the center of gravity of the body
   (b) centroid of the body
   (c) surface of the body
   (d) none of the above [*Ans.* (a)]
5. Parallelogram law of forces states that if two forces acting simultaneously at a point be represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represented in magnitude and direction by
   (a) longer side of the other two sides
   (b) shorter side of the other two sides
   (c) diagonal of the parallelogram which passes through their points of intersection
   (d) diagonal of the parallelogram which does not pass through their point of intersection [*Ans.* (c)]
6. The angles between two forces to make their resultant a minimum and a maximum respectively are
   (a) 0° and 90° (b) 180° and 90°
   (c) 90° and 180° (d) 180° and 0° [*Ans.* (d)]
7. If two forces P and Q ($P > Q$) act on the same straight line but in opposite direction, their resultant is
   (a) $P + Q$ (b) P/Q (c) $Q - P$ (d) $P - Q$
   [*Ans.* (d)]
8. If two forces, acting at a point are represented by two sides of a triangle taken in order, select the condition that is satisfied
   (a) The magnitude of the resultant is zero
   (b) The third side taken in the same order represents their resultant
   (c) The magnitude of the resultant is maximum
   (d) The third side taken in the reverse order represents their resultant
   [*Ans.* (d)]
9. If two equal forces of magnitude P act at an angle θ, their resultant, will be
   (a) $2P \cos \theta/2$ (b) $P \tan \theta/2$ (c) $2P \sin \theta/2$ (d) $P \cos \theta/2$
   [*Ans.* (a)]

10. Two forces can be in equilibrium only if they are
I. equal in magnitude II. opposite in direction III. collinear in action
(a) I and II only (b) I and III only (c) II and III only (d) I, II and III
[*Ans.* (d)]

11. The action of a given system of forces on a rigid body will in no way be changed if we
I. Add to them another system of forces in equilibrium
II. Subtract from them another system of forces in equilibrium
(a) I only (b) II only (c) I and II (d) none of these
[*Ans.* (c)]

12. Consider the following statements:
The principle of superposition is applied to
I. linear elastic bodies
II. bodies subjected to small deformations
Of these statements
(a) I alone is correct (b) I and II are correct
(c) II alone is correct (d) Neither I nor II is correct
[*Ans.* (b)]

13. Match the list I with list II and select the correct answer using the codes given below the lists:

List I	*List II*
A. Co-planar forces	1. Lines of action of all forces lie in the same plane but do not pass through a common point.
B. Concurrent forces	2. Lines of action of all forces lie in the same plane and pass through a common point.
C. Concurrent coplanar forces	3. Lines of action of all forces lie in the same plane.
D. Collinear forces	4. Lines of action of all forces pass through a common point.
	5. Lines of action of all forces lie along the same line.

Codes:
(a) A – 5 B – 4 C – 1 D – 3
(b) A – 4 B – 3 C – 2 D – 1
(c) A – 4 B – 5 C – 2 D – 1
(d) A – 3 B – 4 C – 2 D – 5 [*Ans.* (d)]

14. Match List I with List II and select the correct answer using the codes given below the lists:

List I	*List II*
A. Triangle law of forces	1. Two forces can be in equilibrium, if they are equal in magnitude, opposite in direction and collinear in action.
B. Law of equilibrium of two forces	2. The action of a given system of forces on a rigid body will no way be changed if we add to or subtract from them another system of forces in equilibrium.
C. Law of superposition	3. If two forces acting simultanously on a particle represented by the two sides of a triangle (in magnitude and direction) taken in order then their resultant is represented by the third side (closing side) taken in an opposite order.

D. Theorem of transmissibility of a force	4. The point of application of a force may be transmitted along its line of action without changing the effect of the force on any rigid body to which it may be applied.

Codes:

(a) A – 1 B – 2 C – 3 D – 4

(b) A – 2 B – 1 C – 3 D – 4

(c) A – 3 B – 1 C – 2 D – 4

(d) A – 3 B – 1 C – 4 D – 2 [*Ans.* (c)]

15. Assertion (A): The theorem of transmissibility states that the point of application of a force may be transmitted along its line of action without changing the effects of the force on any rigid body to which it may be applied.

Reason (R): The theorem of transmissibility of a force is limited to those problems of statics in which we are interested only in the conditions of equilibrium of a rigid body and not with the internal forces to which it is subjected.

Select your answer using the codes given below and mark your answer accordingly.

Codes:

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not a correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true. [*Ans.* (b)]

16. A ball of weight W is supported on smooth planes as shown in the given Fig. A.

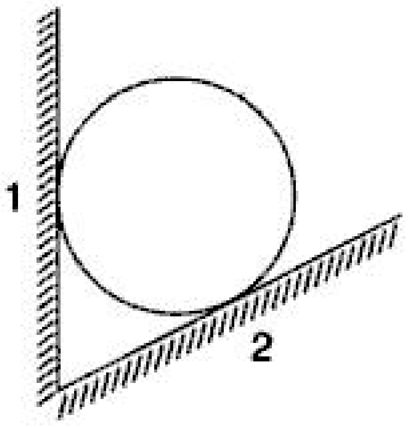
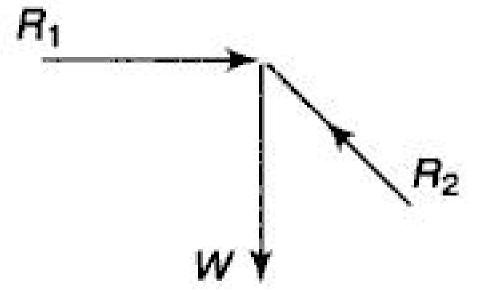
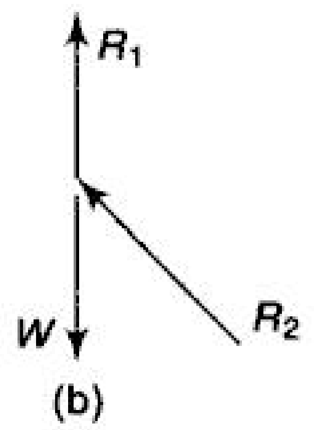


Fig. A

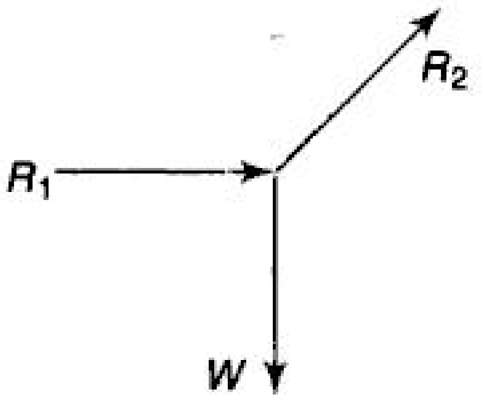
The free body diagram of the ball will be



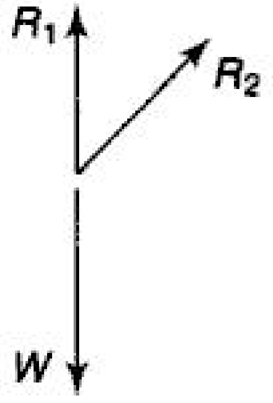
(a)



(b)



(c)



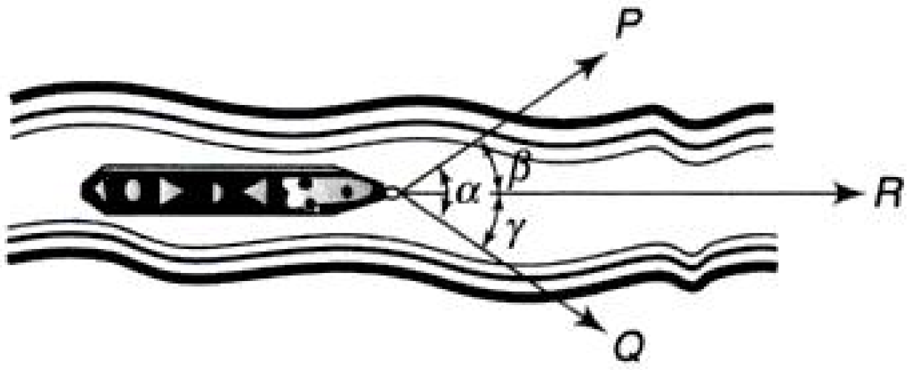
(d)

[*Ans.* (a)]

PROBLEM SET 2.1

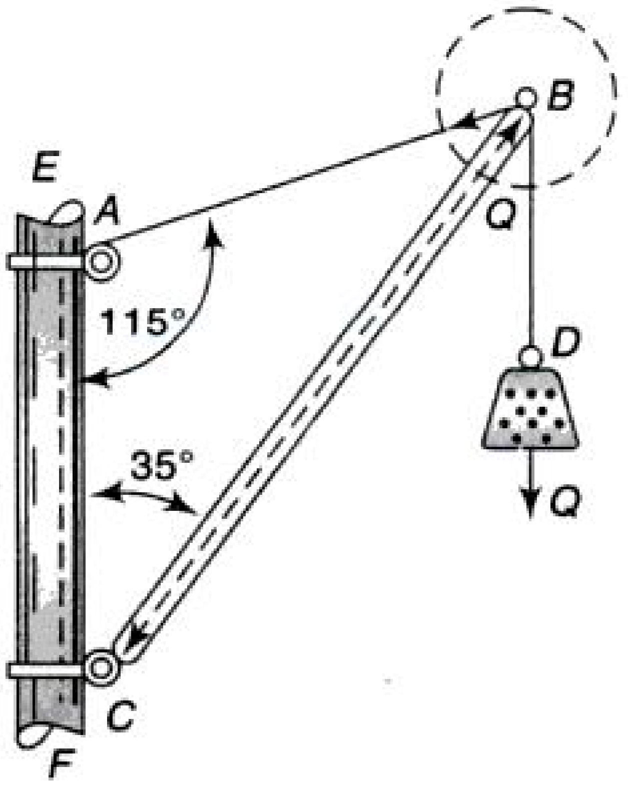
1. A man of weight $W = 712$ N holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight $Q = 534$ N. Find the force with which the man's feet press against the floor. (*Ans.* 178 N)
2. A boat is moved uniformly along a canal by two horses pulling with forces $P = 890$ N and $Q = 1068$ N acting under an angle $\alpha = 60°$ (Fig. A). Determine the magnitude of the resultant pull on the boat and the angles β and γ as shown in the figure. (*Ans.* R=1698 N; $\beta = 33°$; $\gamma = 27°$)

Fig. A

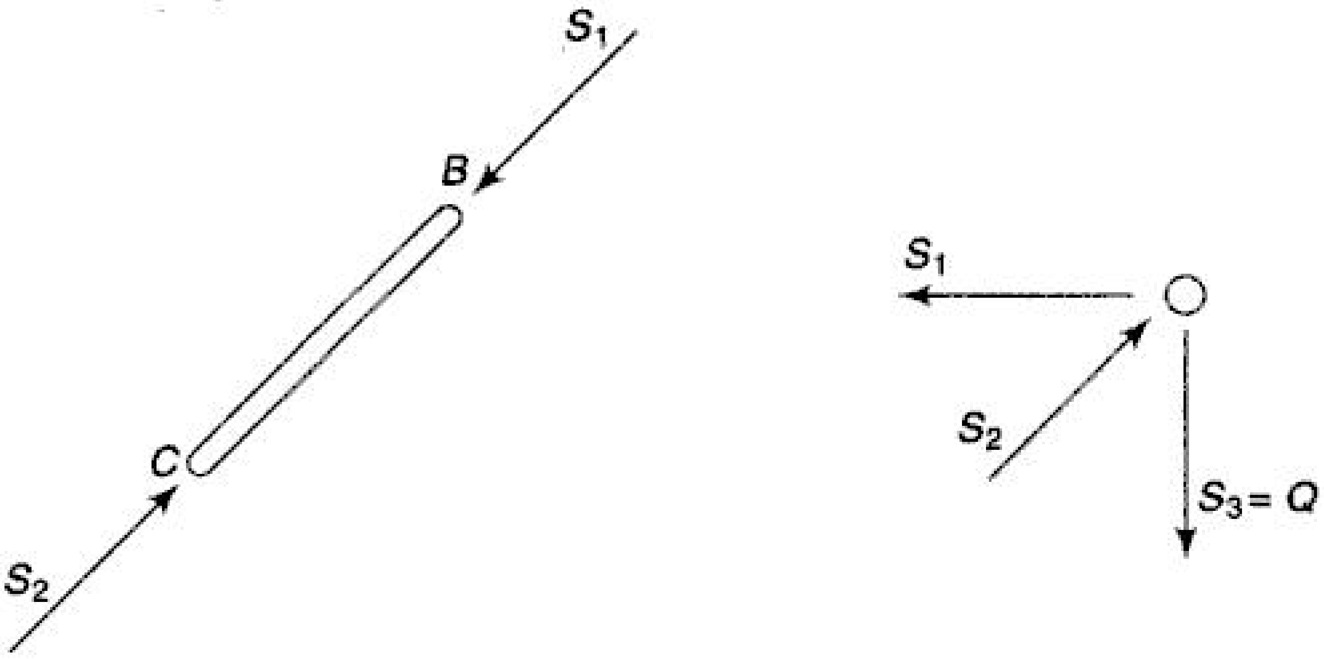


3. What force Q combined with a vertical pull $P = 27$ N will give a horizontal resultant force $R = 36$ N? (*Ans.* 45 N inclined by 36° 52′)
4. To move a boat uniformly along a canal at a given speed requires a resultant force $R = 1780$ N. This is accomplished by two horses pulling with forces P and Q on two ropes, as shown in Fig. A. If the angles that the two ropes make with the axis of the canal are $\beta = 35°$ and $\gamma = 25°$, what are the corresponding tensions in the ropes? (*Ans.* $P = 868$ N; $Q = 1179$ N)
5. If, in Fig. A, the horses pull with the forces $P = 1068$ N and $Q = 890$ N, what must be the angles β and γ to give the resultant $R = 1780$ N ? (*Ans.* $\beta = 22° 22'$; $\gamma = 27° 12'$)
6. Draw the free body diagram of the boom BC and point B shown in Fig. B.

Fig. B

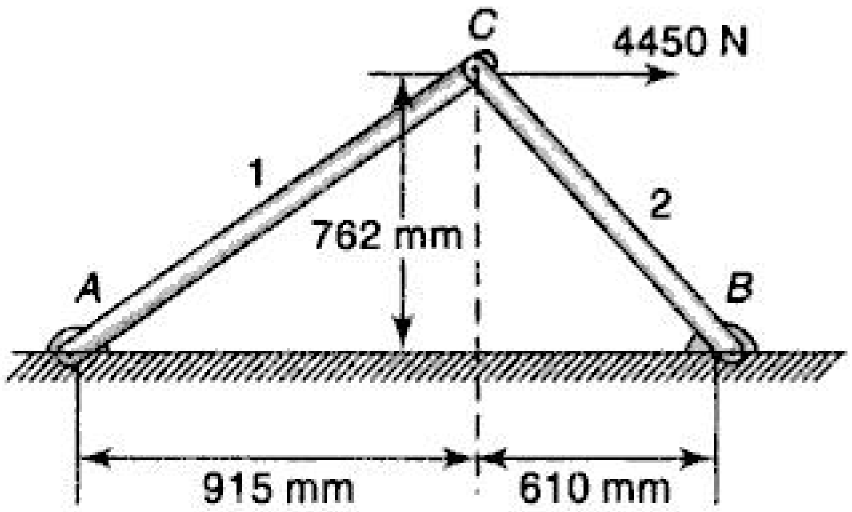


Ans.



S_1
B
C
S_2
S_1
S_2
$S_3 = Q$

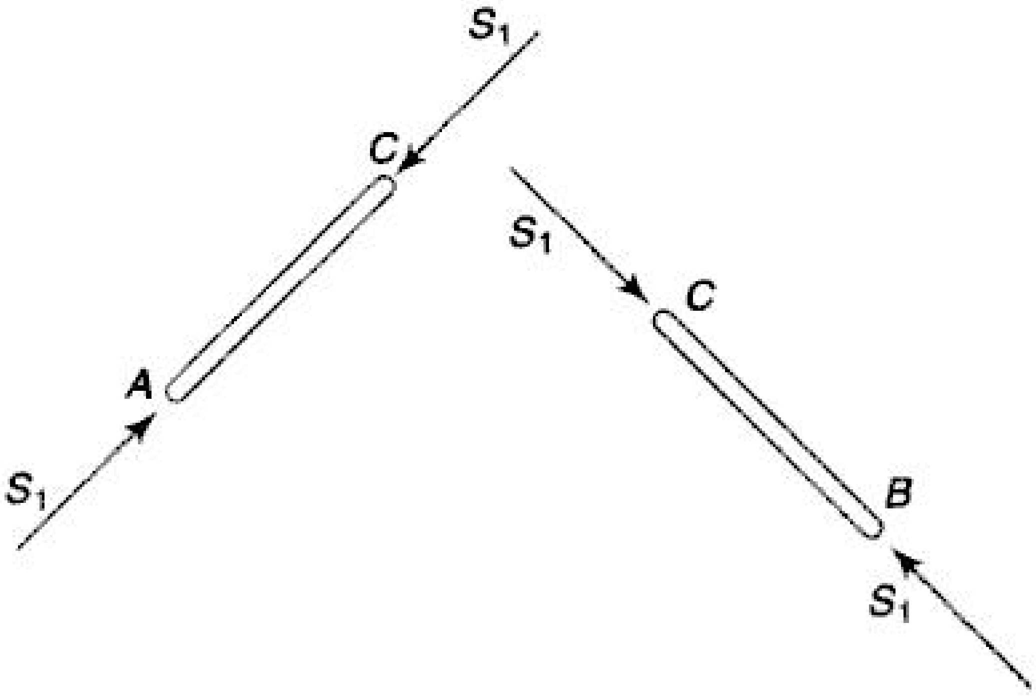
7. Draw the free body diagram of the bars AC and BC shown in Fig. C.



C
4450 N
1
2
762 mm
A
B
915 mm
610 mm

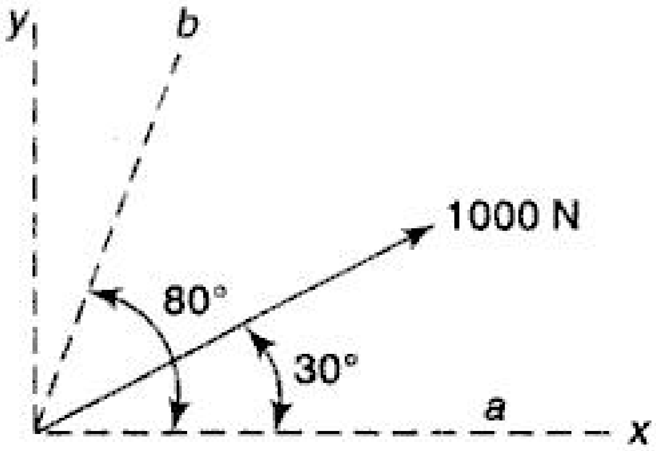
Fig. C

Ans.



S_1
C
S_1
C
A
S_1
B
S_1

8. Determine the components of the 1000 N force along the axes a and b as shown in Fig. D. Compare these components with the rectangular x-and y-components.
(*Ans.* $F_a = 777.86$ N, $F_b = 507.71$ N; $F_x = 866$ N and $F_y = 500$ N)



y
b
1000 N
80°
30°
a
x

Fig. D

9. Resolve the force into rectangular components with the given co-ordinate axes as shown in Fig. E.

(*Ans.* (a) $F_x = -F \sin \alpha, F_y = F \cos \alpha$
(b) $F_x = F \sin \alpha, F_y = F \cos \alpha$
(c) $F_x = F \sin \alpha; F_y = -F \cos \alpha$
(d) $F_x = F \cos (\alpha - \beta); F_y = F \sin (\alpha - \beta)$

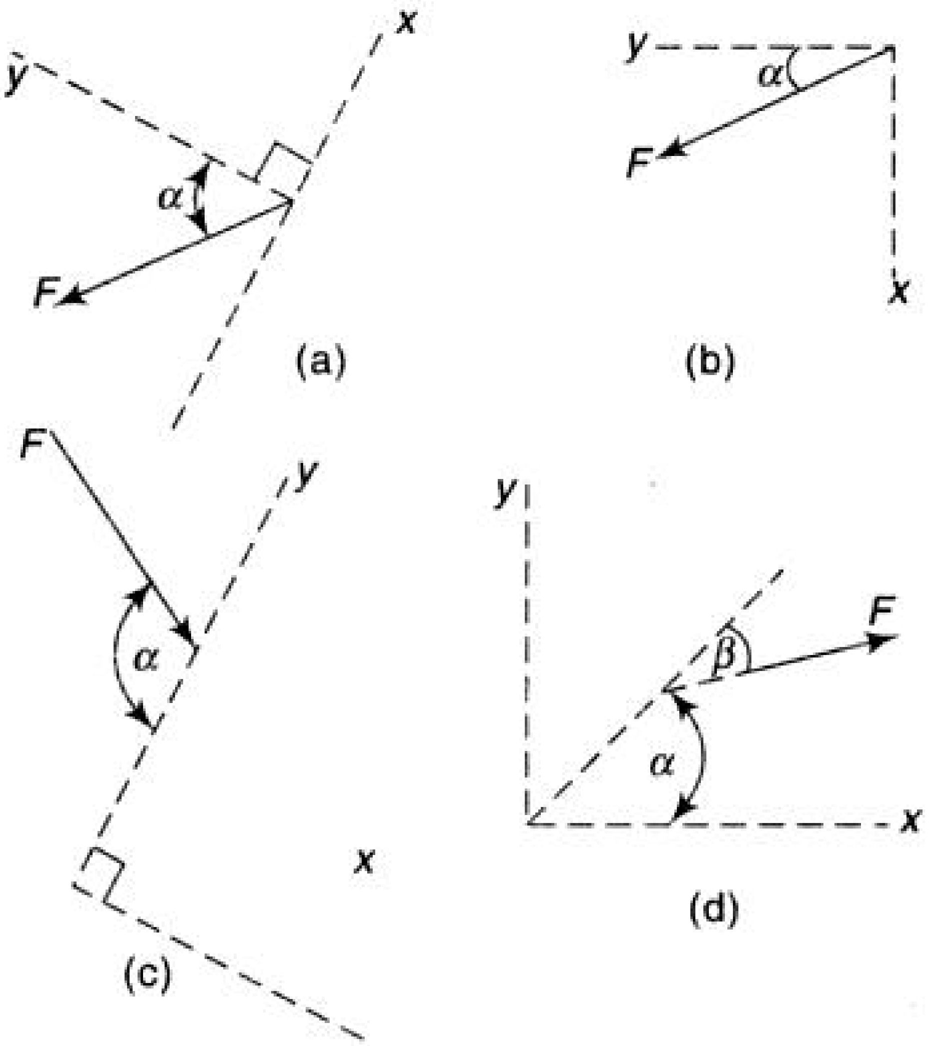


Fig. E

10. In level flight, the chord *AB* of an airplane wing makes an angle $\alpha = 5°$ with the horizontal (Fig. F). The resultant wind pressure on the wing for such conditions is defined by its lift and drag components $L = 6675$ N and $D = 890$ N, which are vertical and horizontal, respectively, as shown. Resolve this force into rectangular components *X* and *Y*, coinciding with the chord *AB* and its normal, respectively.

(*Ans.* X=304.8 N; Y = 6727 N)

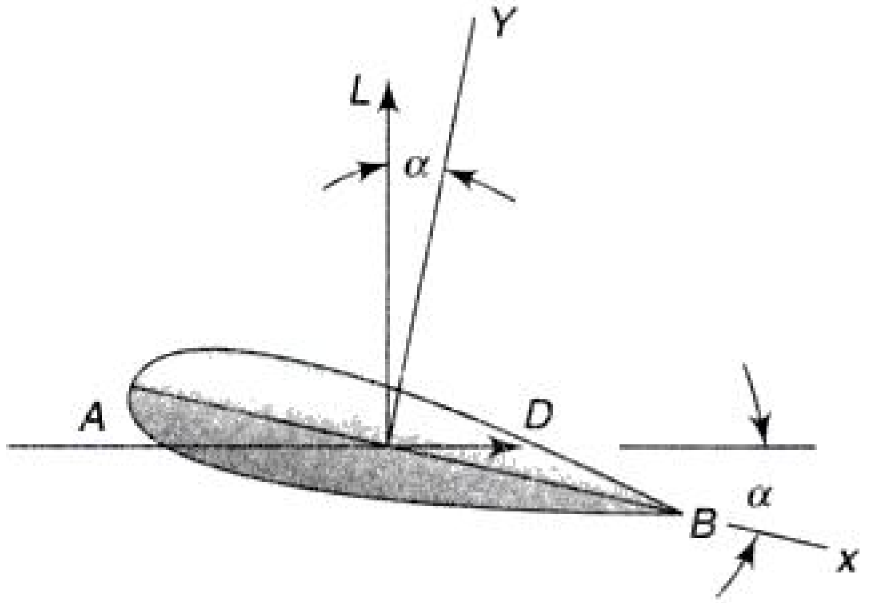


Fig. F

11. A small block of weight $Q = 44.5$ N is placed on an inclined plane which makes an angle $\alpha = 30°$ with the horizontal. Resolve the gravity force Q into two rectangular components Q_t and Q_n acting parallel and normal, respectively, to the inclined plane.

(*Ans.* $Q_t = 22.25$ N; $Q_n = 38.54$ N)

12. For the particular position shown in Fig. G, the connecting rod BA of an engine exerts a force $P = 2225$ N on the crank pin at A. Resolve this force into two rectangular components P_h and P_v acting horizontally and vertically, respectively, at A.
(*Ans.* $P_h = 2081.4$ N; $P_v = 786.5$ N)

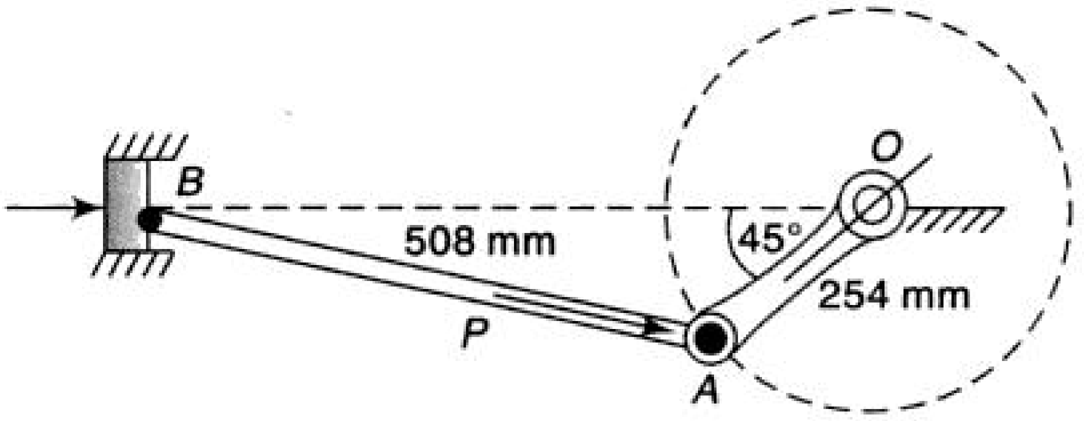


Fig. G

13. Resolve the force P in Fig. G into two rectangular components P_r and P_t acting along the radius AO and perpendicular thereto, respectively.
(*Ans.* $P_r = 915.6$ N; $P_t = 2028$ N)
14. If the resultant of two forces exerted on the body at A of Fig. H is to be vertical, determine the value of β for which the magnitude of P is maximum, and the corresponding magnitude of P. (*Ans.* $\beta = 15°$; $P = 3732$ N)

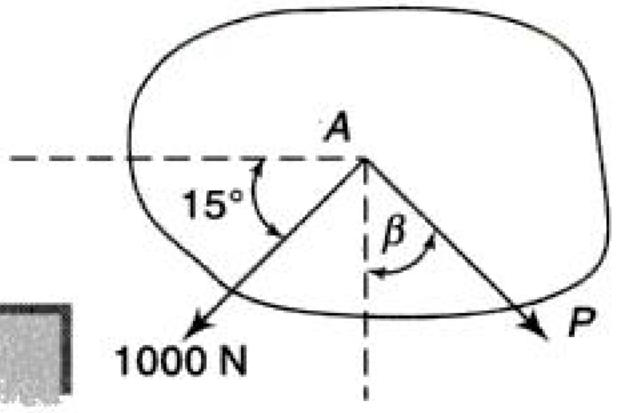


Fig. H

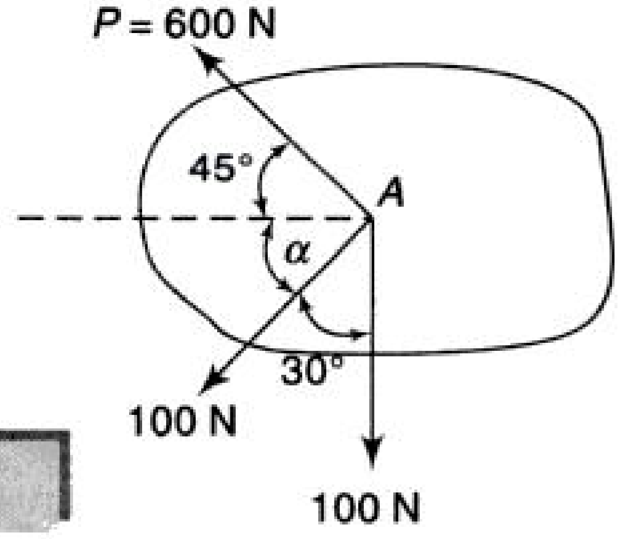


Fig. I

15. Three forces are applied to a body as shown in Fig. I. The direction of the two 100 N may vary, but the angle between these forces is always 30°. Determine the range of values of θ for which the magnitude of the resultant of the forces applied to the body is less than 650 N. (*Ans.* 23.76° to 216.24°)
16. Three forces are applied to a body as shown in Fig. I. The direction of the 100 N forces may vary, but the angle between the forces is always 30°. Determine the value of α for which the resultant of the forces acting at A is directed horizontally to the left if $P = 250$ N. (*Ans.* 51.205°)

2.2 COMPOSITION AND RESOLUTION OF FORCES

Composition

The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces*. If several forces F_1, F_2, F_3, ……, applied to a body at one point, all act in the same plane, they represent a system of forces that can be reduced to a single resultant force. It then becomes possible to find this resultant by successive applications of the parallelogram law. Let us consider, for example, the four forces F_1, F_2, F_3 and F_4 acting on a body at point A [Fig. 2.31(a)]. To find their resultant, we begin by obtaining the resultant $\overline{AC}$ of the two forces F_1 and F_2. Combining this resultant with the force F_3, we obtain the resultant $\overline{AD}$ which must be equivalent to F_1, F_2 and F_3. Finally, combining the forces $\overline{AD}$ and F_4, we obtain the resultant R of the given system F_1, F_2, F_3, F_4. This procedure may be carried on for any number of given forces acting at one point in a plane.

It is evident, in the above case, that exactly the same resultant R will be obtained by successive geometric addition of the free vectors representing the given forces [Fig. 2.31(b)]. In this case we begin with the vector $\overline{AB}$ representing the force F_1. From the end B of this vector we construct the vector $\overline{BC}$, representing the force F_2, and afterward, the vectors $\overline{CD}$ and $\overline{DE}$, representing the forces F_3 and F_4. The polygon $ABCDE$ obtained in this way is the same as the polygon $ABCDE$ in Fig. 2.31(a), and the vector $\overline{AE}$, from the beginning A of the vector

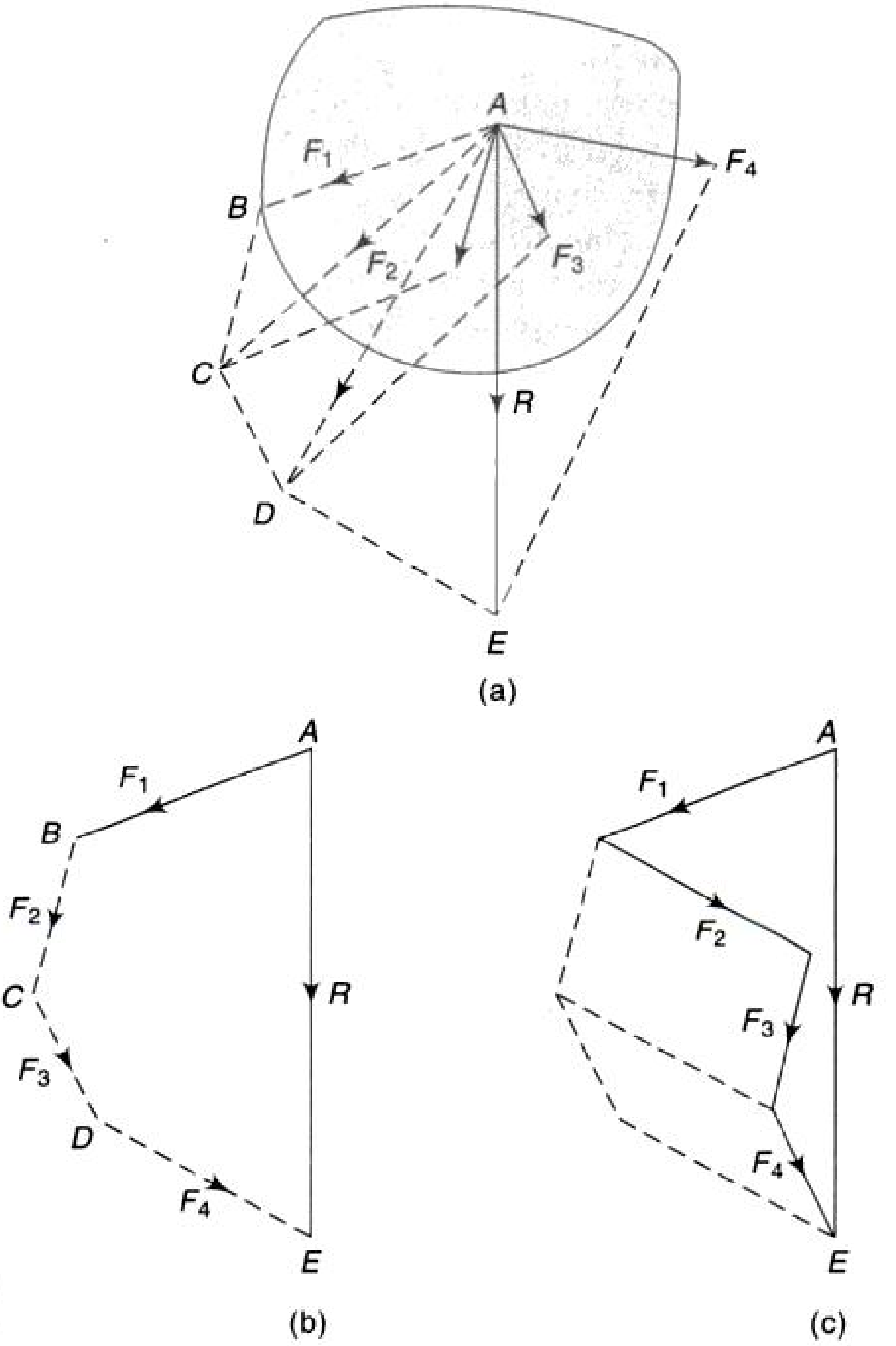


Fig. 2.31

$\overline{AB}$ to the end E of the vector $\overline{DE}$, gives the resultant R which, of course, must be applied at point A in Fig. 2.31(a). The polygon $ABCDE$ in Fig. 2.31(b) is called the *polygon of forces* and the resultant is given by the *closing side* of this polygon. It is always directed from the beginning of the first vector to the end of the last vector. Thus, we may say that the resultant of any system of concurrent forces in a plane is obtained as the geometric sum of the given forces. The construction of the polygon of forces, for determining the resultant, is much more direct for a large number of forces than successive applications of the parallelogram law and is preferable in the solution of problems.

It is evident that the resultant R will not depend upon the order in which the free vectors representing the given forces are geometrically added. For instance, in the above example, we can begin with the force F_1, add to it the force F_4 and afterward the forces F_2 and F_3. Proceeding in this way the polygon of forces shown in Fig. 2.31(c) will be obtained. The closing side $\overline{AE}$ of the polygon gives the same resultant R as before.

In the particular case where the given forces are all acting along one line, the sides of the polygon of forces will all lie along one line and the geometric summation will be replaced by an algebraic summation. The resultant, in this case, is the algebraic sum of its components.

If the end of the last vector coincides with the beginning of the first, the resultant R is equal to zero and the given system of forces is in equilibrium.

Resolution of a Force

The replacement of a single force by several components, which will be equivalent in action to the given force, is called the problem of *resolution of a force*. The case in which a single force is to be replaced by two components is the one most commonly encountered already discussed in the previous Section.

Regarding the resolution of a given force into three coplanar components, acting in three given directions, we see that the magnitude of one of the components can be arbitrarily chosen so that in this case the problem is entirely indeterminate. In the general case of resolution of a force into any number of coplanar components intersecting at one point on its line of action, the problem will be indeterminate unless all but two of the components are completely specified as to both magnitude and direction.

Important Terms and Concepts

Composition of forces	Resultant	Polygon of forces	Equilibrium
Resolution of a force	Rectangular components		

SUMMARY

- Composition of forces: The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces.*
- Polygon of forces: If a number of coplanar concurrent forces are acting on a body such that they can be represented in magnitude and direction by the sides of a polygon taken in an order, their resultant is represented in both magnitude and direction by the closing side of the polygon taken in the opposite order. It is evident

that the resultant R will not depend upon the order in which the free vectors representing the given forces are geometrically added. In the particular case where the given forces are all acting along one line, the sides of the polygon of forces will all lie along one line and the geometric summation will be replaced by an algebraic summation. The resultant, in this case, is the algebraic sum of its components. If the end of the last vector coincides with the beginning of the first, the resultant R is equal to zero and the given system of forces is in equilibrium.

- The replacement of a single force by several components which will be equivalent in action to the given force is called the problem of *resolution of a force.* The case in which a single force is to be replaced by two components is the one most commonly encountered. By using the parallelogram law, we can resolve a given force R into any two components P and Q intersecting at a point on its line of action. We shall discuss two possible cases: (1) The directions of both components are given; their magnitudes, to be determined and (2) both the direction and magnitude of one component are given; the direction and magnitude of the other, to be determined.
- When a force is resolved into two components, which are acting at right angles to each other, they are called *rectangular components.*
- Regarding the resolution of a given force into three coplanar components, acting in three directions, the problem is indeterminate.
- In the general case of resolution of a force into any number of coplanar components intersecting at one point on its line of action, the problem will be indeterminate unless all but two of the components are completely specified as to both magnitude and direction.

PRACTICE SET 2.2

Review Questions

1. Define Composition of Forces.
2. State and explain law of polygon of forces.
3. What is meant by resolution of a force?
4. Define equilibrium.
5. Differentiate between 'composition of forces' and 'resolution of a force'.

Objective Questions

1. Consider the following statements:
   I. The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of *composition of forces.*
   II. The replacement of a single force by several components which will be equivalent in action to the given force is called the problem of *resolution of a force.*
   Of these statements
   (a) I alone is correct
   (b) II alone is correct
   (c) I and II are correct
   (d) Neither I nor II is correct
   [*Ans.* (c)]

2. Consider the following statements:
   I. When a force is resolved into two components, which are acting at right angles to each other, they are called *rectangular components.*
   II. Regarding the resolution of a given force into three coplanar components, acting in three directions, the problem is indeterminate.
   Of these statements
   (a) I alone is correct
   (b) II alone is correct
   (c) I and II are correct
   (d) Neither I nor II is correct
   [*Ans.* (c)]

3. The resultant of two forces is equal to each of the force. The angle between them is
   (a) 0° (b) 90° (c) 180° (d) 120°
   [*Ans.* (d)]

PROBLEM SET 2.2

1. Determine graphically the magnitude and direction of the resultant of the four forces shown in Fig. A. (*Ans.* $R = 1860$ N; $\theta = 61° 45'$)

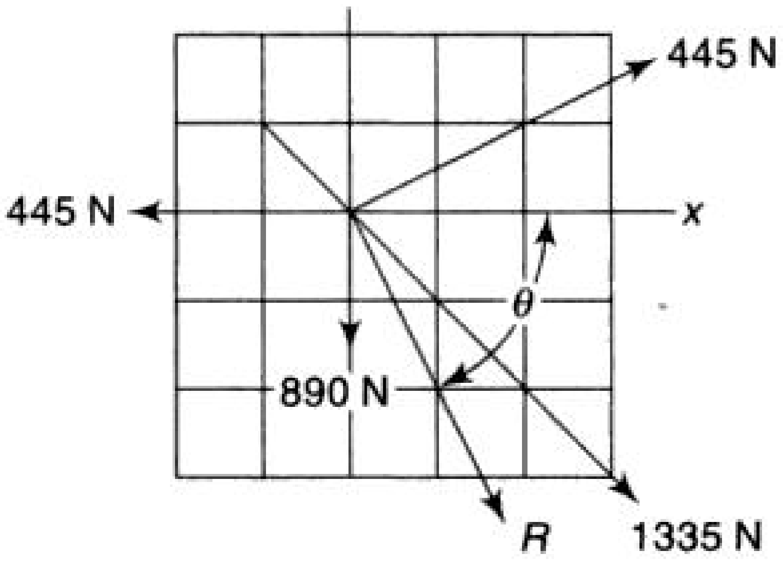


Fig. A

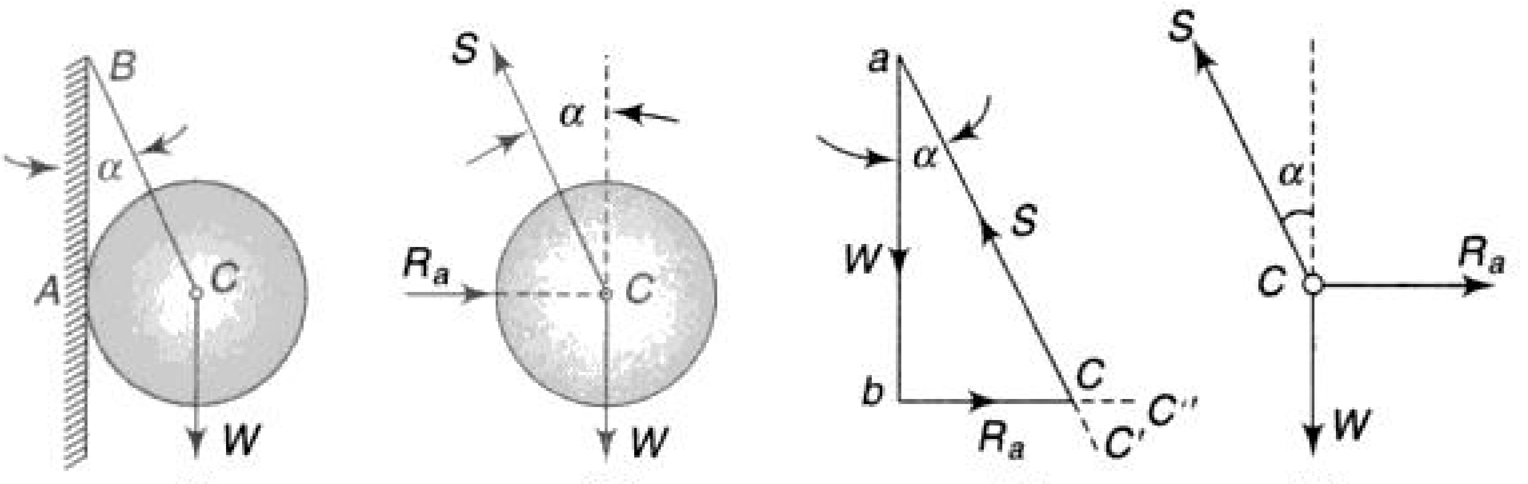
2. Determine graphically the magnitude and direction of the resultant of the four concurrent forces in Fig. A if each of the 445 N forces is increased to 667.5 N. (*Ans.* $R = 1766.5$N; $\theta = 60.36°$)
3. Determine graphically the magnitude and direction of the resultant of the four concurrent forces in Fig. A if each of the 445 N forces is reversed in direction.

2.3 EQUILIBRIUM OF CONCURRENT FORCES IN A PLANE

In Section 2.2, it was shown that the resultant of any number of concurrent forces in a plane is given by the closing side of the polygon of forces obtained by successive geometric addition of their free vectors. In the particular case where the end of the last vector coincides with the beginning of the first, the resultant vanishes and the system is in equilibrium. The reverse of this statement is a very important one: *If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces, or rather their free vectors, when geometrically added must form a closed polygon.* This statement represents the condition of equilibrium for any system of concurrent forces in a plane.

In Fig. 2.32, we again consider the ball supported in a vertical plane by a string BC and a smooth wall AB. The free-body diagram in which the ball has been isolated from its supports and in which all forces acting upon it, both active and reactive, are indicated by vectors is shown in Fig. 2.32(b). The details of making this free-body diagram were already discussed in Section 2.1. The three concurrent forces W and R_a are a system in equilibrium and hence their free vectors must build a closed polygon, in this case, a triangle.

To construct this closed triangle of forces [Fig. 2.32(c)], we first lay out the vector $\overline{ab}$, representing to scale the magnitude of the gravity force W. Then from the ends a and b of this vector, we construct the lines ac' and bc'' parallel, respectively, to the known lines of action of the reactive forces S and R_a. The intersec-



(a) (b) (c) (d)

Fig. 2.32

tion C of these two lines determines the required magnitudes of R_a and S and the arrows show the directions that these forces must have to build a closed triangle. It should be noted that in any closed polygon of forces, the vectors must follow one another tail to head around the polygon. The magnitudes of R_a and S may now be scaled from the drawing and the problem is solved.

The foregoing procedure, in which the closed polygon of forces is constructed to scale and the magnitudes of the reactions measured directly from the drawing, is called a graphical solution of the problem. In order to make such a solution, of course, numerical values would have to be given for the magnitude of W and α, the angle between the string and the wall.

If numerical data are not given, we can still sketch the closed triangle of forces as shown in Fig. 2.32(c) and then express the magnitudes of R_a and S in terms of W and α by trigonometry. In the present case, for example, we see from the triangle of forces that

$$R_a = W \tan \alpha \qquad S = W \sec \alpha \tag{a}$$

Then for any given numerical data, the magnitudes of R_a and S can be computed from expressions (a). Such an analysis of the problem is called a *trigonometric solution*.

Lami's Theorem: *If three concurrent forces are acting on a body, kept in an equilibrium, then each force is proportional to the sine of the angle between the other two forces and the constant of proportionality is the same.*

Consider forces P, Q and R acting at point O as shown in Fig. 2.33(a). Mathematically, Lami's theorem is given by the following equation,

$$\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma} = k \tag{b}$$

Since the forces are in equilibrium, the triangle of forces should close. Draw the triangle of forces ΔABC as shown in Fig. 2.33(b) corresponding to the forces P, Q and R acting at a point O. The angles of triangle are

$$\angle A = \pi - \alpha$$
$$\angle B = \pi - \beta$$
$$\angle A = \pi - \gamma$$

From the sine rule for the triangle, we get

$$\frac{P}{\sin(\pi - \alpha)} = \frac{Q}{\sin(\pi - \beta)} = \frac{R}{\sin(\pi - \gamma)}$$

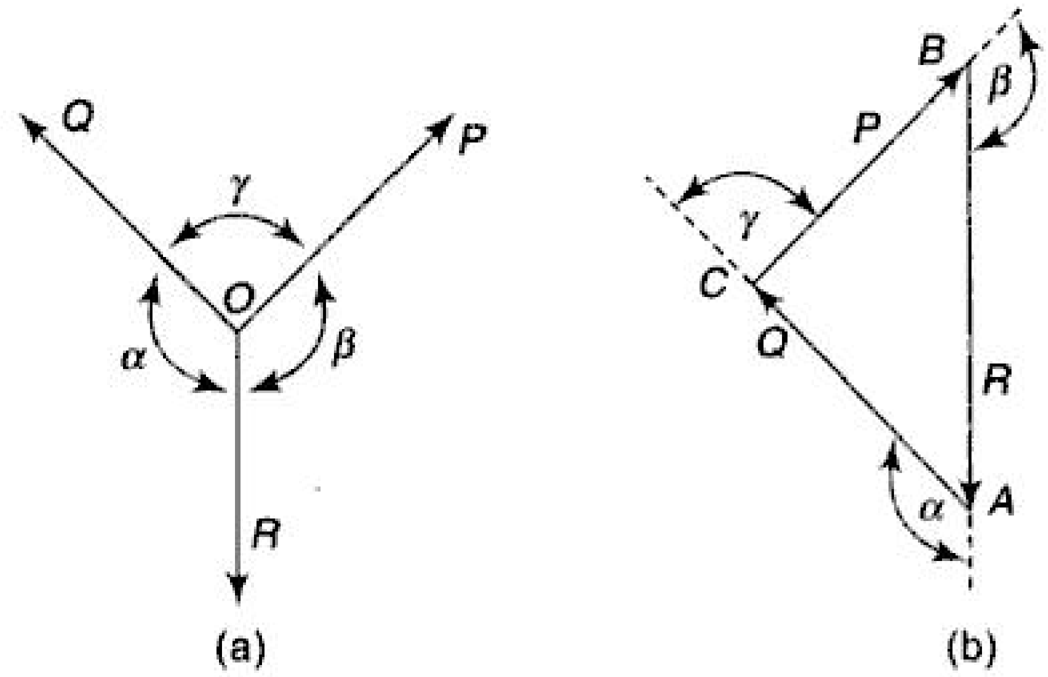


Fig. 2.33 (a) (b)

$$\because \quad \sin(\pi-\alpha)=\sin\alpha$$
$$\sin(\pi-\beta)=\sin\beta$$
$$\sin(\pi-\gamma)=\sin\gamma$$

Substituting these values into the above equation, it reduces to the Lami's theorem, i.e.,

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$

Note ►► While applying Lami's theorem, in the free-body diagram of the body, draw the direction of forces either directed towards or away from the point of concurrency.

Returning to the free-body diagram of Fig. 2.32(d), using Lami's theorem, we will get

$$\frac{R_a}{\sin(\pi-\alpha)}=\frac{W}{\sin\left(\frac{\pi}{2}+\alpha\right)}=\frac{S}{\sin(\pi/2)}$$

Simplifying this equation, we will get the same result as before, i.e, Eq. (a).

When feasible, the trigonometric solution or Lami's theorem is preferable to the graphical solution since it is free from the unavoidable small errors associated with graphical constructions and scaling. However, in more complicated problems, the trigonometric method or Lami's theorem often becomes too involved to be practicable, and we must be satisfied with the less elegant but more straightforward graphical method.

Returning to the free-body diagram in Fig. 2.32(b), we see that to balance the applied gravity force, we need simply an equal, opposite and collinear force which is called the *equilibrant* of the active forces. We may now consider this equilibrant as represented by the vector in Fig. 2.32(c) and proceed to resolve it into components R_a and S parallel to the known lines of action of the reactions. In this way, we get the same magnitudes for R_a and S as before. Recalling that the resolution of a given force into more than two coplanar components is an indeterminate problem [see Eq. (a)], we conclude that in dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we cannot determine definitely the magnitudes of more than two reactive forces.

Suppose, for example, that the ball, otherwise constrained as in Fig. 2.32(a), also rests on a horizontal floor at D, as shown in Fig. 2.34(a). In such case, the

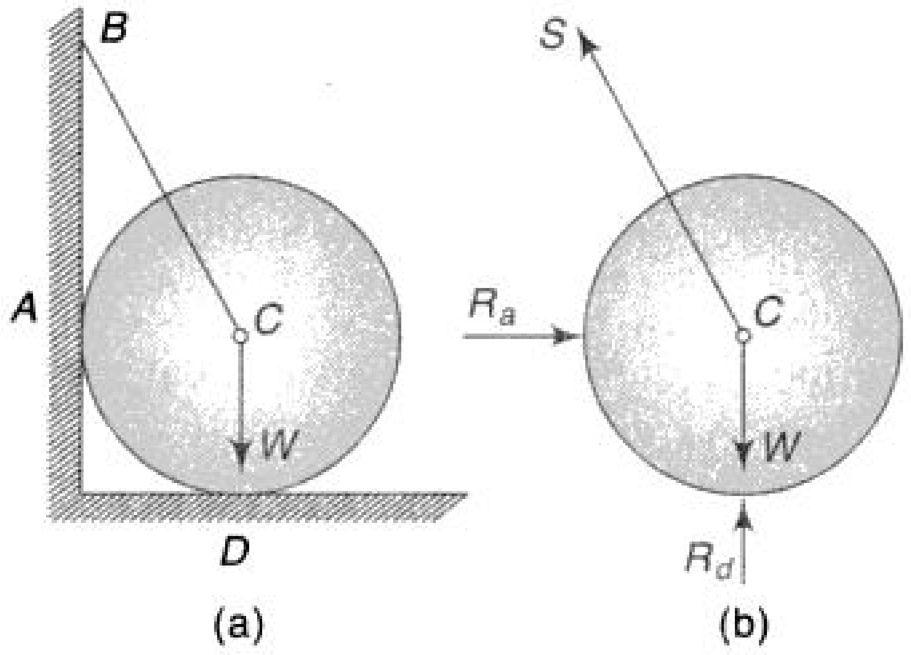


Fig. 2.34 (a) (b)

free-body diagram of the ball will be as shown in Fig. 2.34(b), and we have three reactive forces R_a, S and R_d with unknown magnitudes. While the resultant of these three forces must clearly be the equilibrant of W, there is no way to determine their magnitudes definitely and the problem is said to be statically *indeterminate*. Supports in excess of those necessary and sufficient to completely constrain the ball in the plane of the figure are called *redundant constraints*.

Examples Examples Examples Examples Examples

1. An electric street lamp is suspended from a small ring B supported by two wires AB and CB, the ends A and C of which are on the same level [Fig. 2.35(a)]. Assuming these wires to be perfectly flexible and neglecting their weights, find the force produced in each if the weight of the lamp is 67 N, length of each wire, 3.05 m, and the sag DB, 1.22 m.

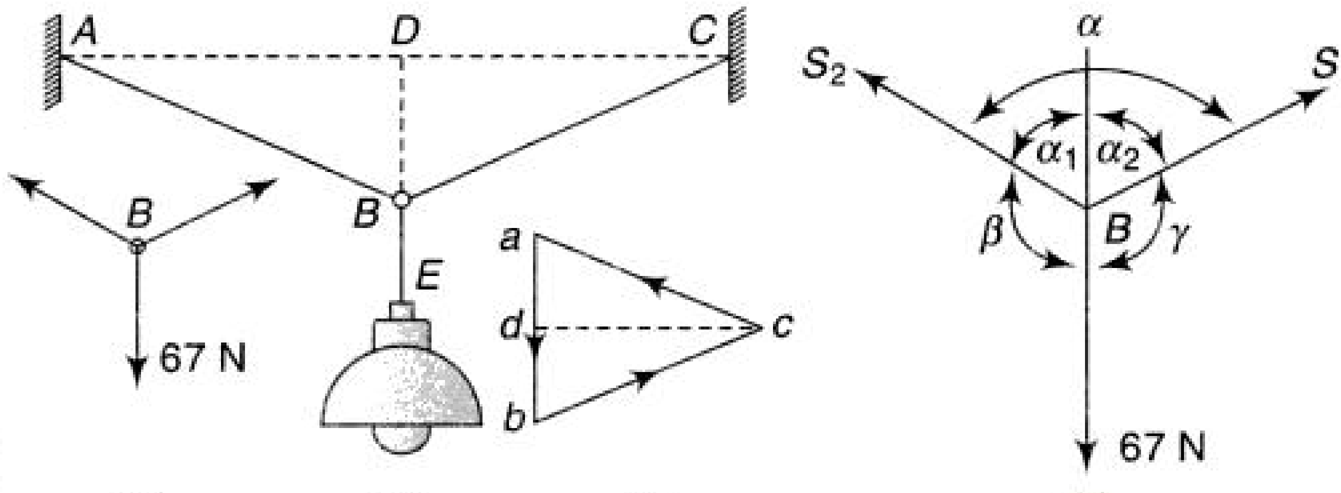


Fig. 2.35 (b) (a) (c) (d)

Solution: Under the action of the gravity force of the lamp, the wire EB pulls down on the ring B which, in turn, exerts a pull on each of the two wires BA and BC. Hence each of these wires is in tension and exerts an equal and opposite reaction on the ring B, the direction of which must coincide with the axis of the wire. Thus the ring B, considered as a free body, is acted upon by three forces as shown in Fig. 2.35(b). Since these three forces are in equilibrium, the vectors representing them must build a closed triangle. To construct this triangle [Fig. 2.35(c)], we begin with the known vector $\overline{ab}$ representing, to a certain scale, the weight of the lamp, and then draw the sides $\overline{bc}$ and $\overline{ca}$ parallel, respectively, to the wires CB and AB.

The lengths of these vectors give the magnitudes of the reactions exerted on the ring *B* by the wires and consequently the magnitudes of the tensile forces in these wires. If the triangle of forces is constructed to scale, the magnitudes of these forces are obtained by scaling the lengths of the vectors $\overline{bc}$ and $\overline{ca}$.

The same magnitudes can be obtained also by calculation. Since the vectors $\overline{bc}$ and $\overline{ca}$, by construction, are parallel, respectively, to the wires *CB* and *AB*, we have ΔDBC similar to Δdbc; from which
$ab : bc = 2BD : BC = 2.44 : 3.05$

and since the force $\overline{ab} = 67$ N, we find $\overline{bc} = \dfrac{3.05}{2.44} \times 67 = 83.75$ N.

Alternate Solution: Drawing the free body diagram of ring *B* as shown in Fig. 2.35(b) is same as before. Figure 2.35(b) is redrawn as shown in Fig. 2.35(d) with the mathamatical notation. The ring *B* is acted upon by three forces as shown in Fig. 2.35(d). Since these three forces are in equilibrium, using Fig. 2.35(d) and applying Lami's theorem, we get

$$\frac{S_1}{\sin\beta} = \frac{S_2}{\sin\gamma} = \frac{67}{\sin\alpha} \qquad \text{(c)}$$

where S_1 = Tension in the wire *AB*
S_2 = Tension in the wire *CB*
$\because \quad \alpha = \alpha_1 = \alpha_2$
and $\quad \alpha_1 = \alpha_2$

$$\cos\alpha_1 = \cos\alpha_2 = \frac{BD}{BC} = \frac{1.22}{3.05}$$

$\therefore \quad \alpha = 2\alpha_1$
$\beta = \pi - \alpha_1 = \gamma$

Substituting these values into the Eq. (c), we obtain

$$S_1 = S_2 = 67\frac{\sin(\pi - \alpha_1)}{\sin\alpha} = 67\frac{\sin\alpha_1}{\sin 2\alpha_1} = 67\frac{\sin\alpha_1}{2\sin\alpha_1\cos\alpha_1} = \frac{67}{2\cos\alpha_1}$$

$$\Rightarrow S_1 = S_2 = \frac{67}{2\times\dfrac{1.22}{3.05}} = = 83.75\,\text{N}$$

So, we got the tensions in the wires *CB* and *AB* same as above.

2. A weight $Q = 2225$ N hanging on a cable *BD* is supported at point *B* by a cable *AB* and a boom *BC* which is hinged at *C* [Fig. 2.36(a)]. Neglecting the weights of the cable and boom and assuming an ideal hinge at *C*, determine the forces transmitted to the mast at points *A* and *C*. The angles of ΔABC are indicated in the figure.

   Solution: We begin by considering the equilibrium of the pin at *B*. The forces acting on this pin are the active vertical gravity force *Q*, acting through the cable *BD*, and the reactions exerted by the cable *AB* and by the boom

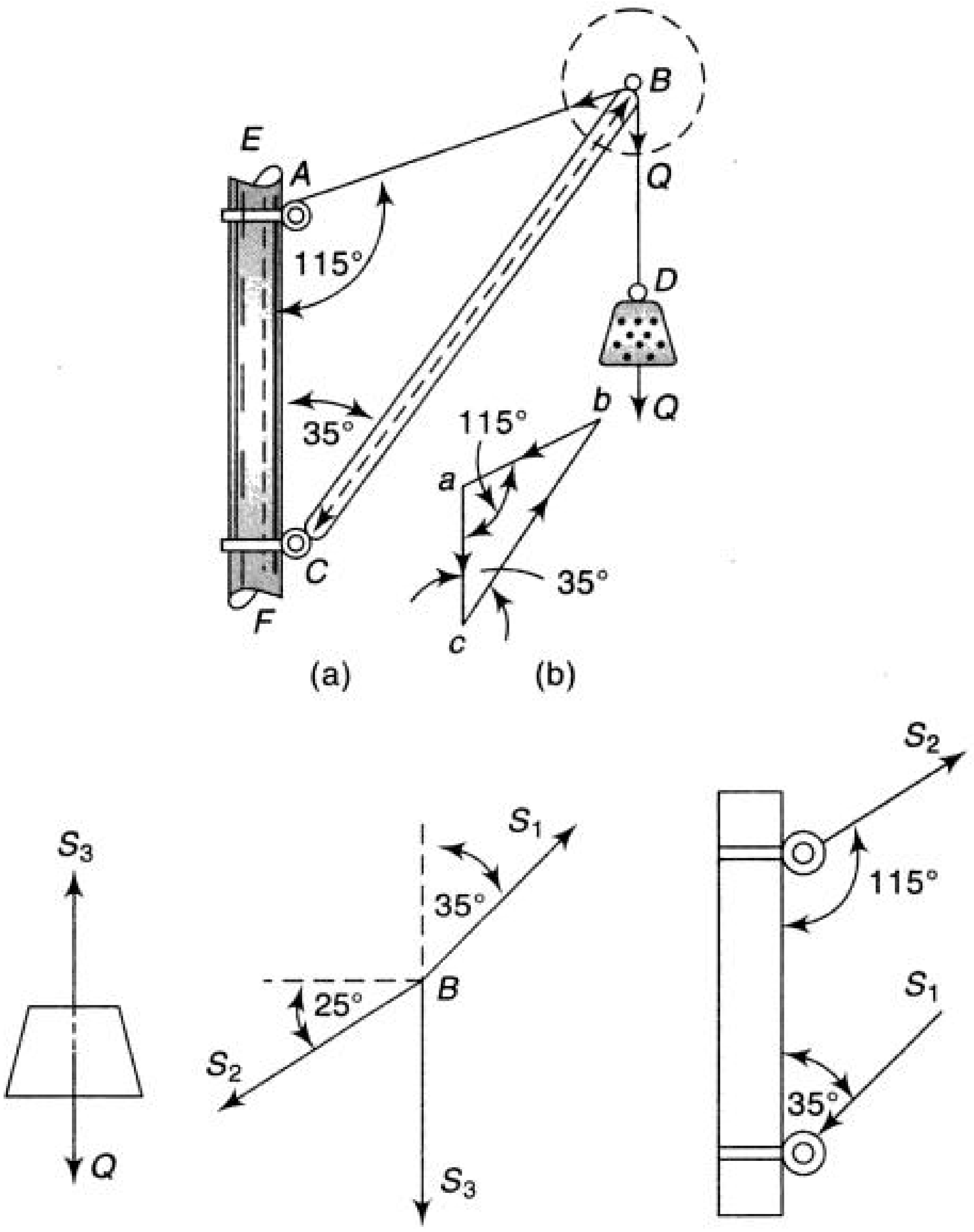


Fig. 2.36 (a) (b) (c) (d) (e)

BC. Since each of these members is a body acted upon by forces only at its ends and since we are assuming an ideal hinge at *C*, we conclude that the direction of each of these reactions must coincide with the axis of the member that produces it. The free-body diagram for the pin at *B* is as shown in the circle around this joint [Fig. 2.36(a)].

Now having given the magnitude and direction of one of the three forces in equilibrium and the lines of action of the other two, the magnitudes of these latter two forces are obtained by constructing the triangle of forces [Fig. 2.36(b)]. Knowing that the vector $\overline{ac}$, representing the weight Q, acts downward, the arrows on the other two vectors $\overline{cb}$ and $\overline{ba}$ must be directed as shown on the triangle of forces, since all arrows must follow each other tail to head around any closed polygon of forces. Considering the vector $\overline{cb}$ which represents the reaction of the boom on the pin at *B*, we see that the boom pushes against this pin and hence is in compression. Similarly, the arrow on the vector $\overline{ba}$ indicates tension in the cable *AB*. In general, if the directions of any unknown reactions are assumed incorrectly in the free-body diagram, they may be corrected after the construction of the polygon of forces.

We conclude now, since the cable *AB* is in tension and the boom *BC* in compression, that the cable pulls on the mast at *A* with a force equal but opposite to the vector $\overline{ba}$ and that the boom pushes on the mast at *C* with a force equal but opposite to the vector $\overline{cb}$. These actions at *A* and *C* are shown in Fig. 2.36(a). Their magnitudes can be obtained either by scaling the lengths of the vectors $\overline{cb}$ and $\overline{ca}$ of the triangle of forces or by trigonometric calculation, which gives, for the force at *A*, 2552.4 N and for the force at *C*, 4033 N.

Alternate Solution: The free body diagram of weight Q in which the body has been isolated from its support and in which all forces acting upon it , both active and reactive, are indicated by vectors is shown in Fig. 2.36(c). Since the body is a two force memeber in equilibrium, applying the law of equilibrium of two forces, we obtain $S_3 = Q$.

Now consider the equilibrium of pin at *B*. The forces acting on this pin are the reactions exerted by the cables *AB* and *BD*, and by the boom *BC*. Since each of these members is a body acted upon by forces only at its ends and since we are assuming an ideal hinge at *C*, we conclude that the direction of each of these reactions must coincide with the axis of the member that produces it.

The direction of S_2 and S_3 is known i.e., *tensile,* since the cables can resist only tensile forces. The reaction from the boom on the pin at *B*, we see that the boom pushes against this pin and hence is in compression. In general, if the directions of any unknown reactions are assumed incorrectly in the free body diagram, they may be corrected after getting negative values for those reactive forces from Lami's theorem. The free body diagram for the pin at *B* is shown in Fig. 2.36(d).

Using free body diagram of pin at *B* shown in Fig. 2.36(d) and applying Lami's theorem, we get

$$\frac{S_1}{\sin\alpha} = \frac{S_2}{\sin\beta} = \frac{Q}{\sin\gamma} \qquad \text{(c)}$$

Substituting the values for $\alpha = 65°$, $\beta = \pi - 35°$ and $\gamma = 150°$ in Eq. (c), we will get

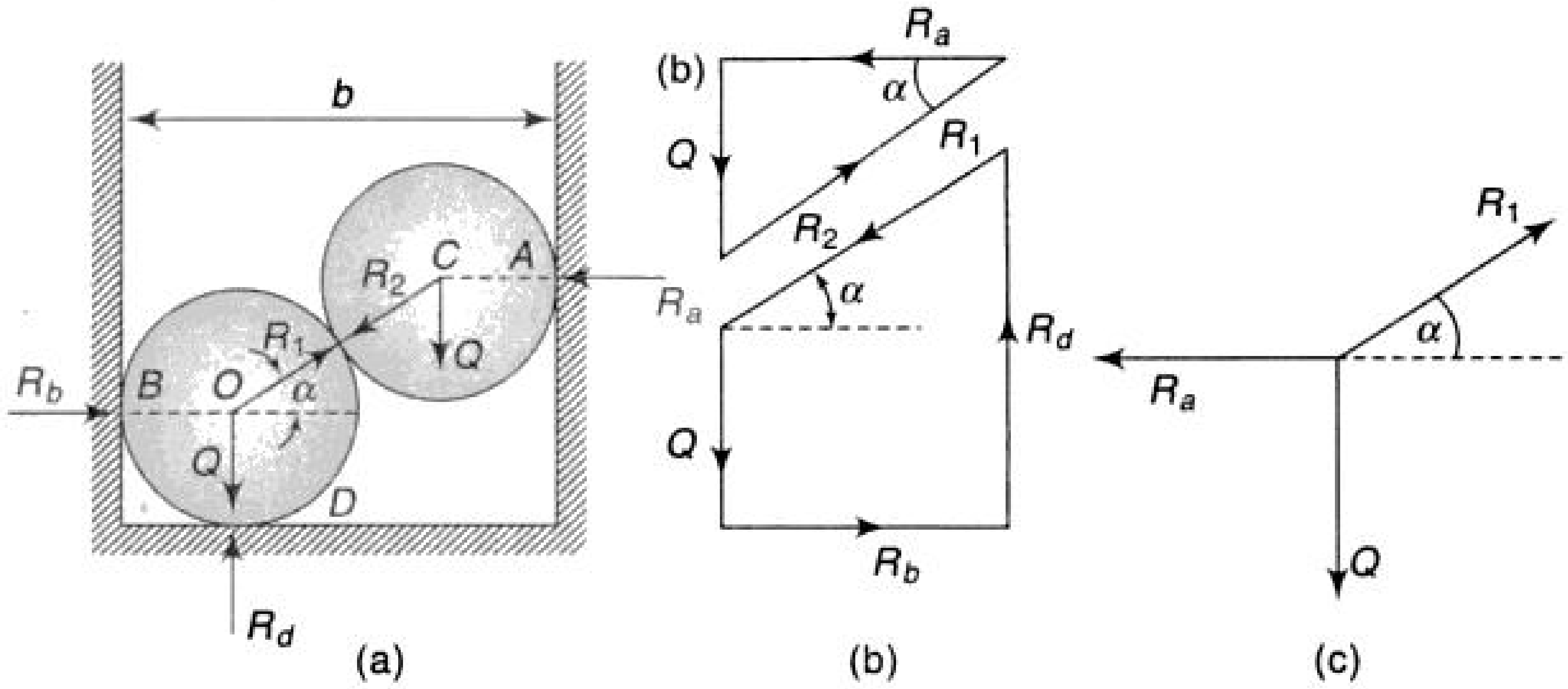
$$S_1 = 4033 \text{ N}$$

and $$S_2 = 252.4 \text{ N}$$

From the law of action and reaction, we will get the forces at points *A* and *C* are 2552.4 N and 4033 N, respectively, and the free body diagram of mast is shown in Fig. 2.36(e).

3. Two smooth spheres, each of radius r and weight Q, rest in a horizontal channel having vertical walls, the distance between them is b (Fig. 2.37(a)). Find the pressures exerted on the walls and floor at the points of contact *A,B* and *D*. The following numerical data are given: r =254 mm b = 914 mm., Q = 445 N.

   Solution: Since the spheres are smooth, the pressures at the various points of contact must be normal to the surfaces. Removing the supporting walls



(a) (b) (c)

Fig. 2.37

and floor and replacing them by their reactions R_a, R_b, and R_d, we obtain the free-body diagram for both spheres as shown in Fig. 2.37(a). These reactions are equal and opposite to the required pressures exerted by the spheres on the walls and floor. At the point of contact between the two spheres, we have two equal and opposite forces R_1 and R_2 and which must act along the line OC joining the centers of the spheres. When considering the equilibrium of the upper sphere, we take only the force R_1 representing the reaction exerted by the lower sphere; likewise when considering the lower sphere, we take only the force R_2. We see now that the upper sphere is in equilibrium under the action of the gravity force Q and the two reactions R_1 and R_a while the lower sphere is in equilibrium under the action of the four forces, R_2, Q, R_b and R_d. In each case all forces are in one plane and concurrent at the center of the corresponding sphere on which they act.

We begin by constructing the triangle of forces for the upper sphere. From this triangle [Fig. 2.37(b)] the reactions R_a and R_1 are determined. Proceeding now to the lower sphere, we have the reaction R_2 equal and opposite to the previously determined force R_1 and the gravity force Q, both of which are completely known. Thus we can complete the polygon of forces [Fig. 2.37(c)] for this sphere and determine the remaining two unknown reactions R_b and R_b.

If the drawing in Fig. 2.37(a) has been made to scale, the direction of the line OC and consequently of the vectors R_1 and R_2 will be determined graphically. In this event, if the polygons of forces have also been constructed to scale, the magnitudes of the various unknown reactions may be scaled directly from the drawings. Otherwise they may be computed as follows: referring to Fig. 2.37(a), we note that from which

$$2r + 2r \cos \alpha = b$$

from which

$$\cos \alpha = \frac{2}{2r} - 1 \tag{d}$$

Using the value of α determined from Eq. (d), we find from the polygons of forces

$$R_a = R_b = 1.33Q = 593.3 \text{ N}$$

and $$R_d = 2Q = 890 \text{ N}.$$

Alternate Solution: Drawing the free body diagram of upper sphere is described above.

Using the free body diagram of the upper sphere as shown in Fig. 2.37(d) and applying Lami's theorem, we get

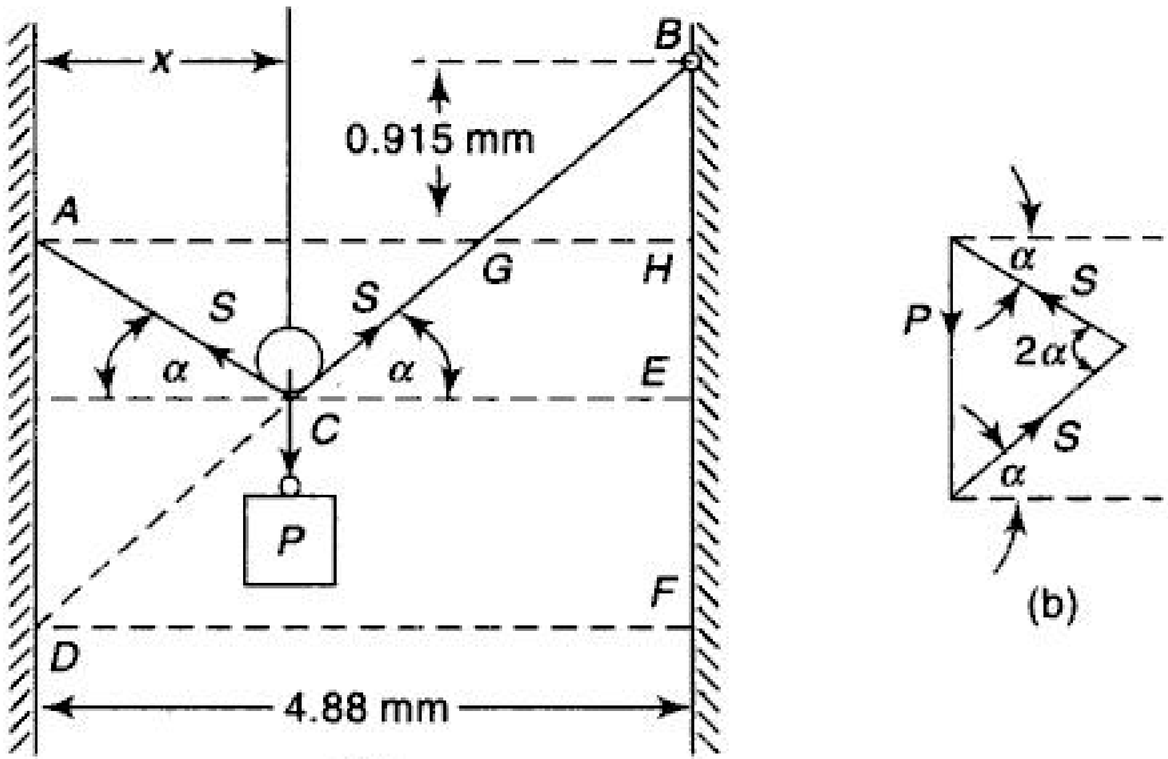
$$\frac{R_1}{\sin \pi/2} = \frac{R_a}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{Q}{\sin(\pi - \alpha)}$$

$$\Rightarrow \qquad R_1 = \frac{Q}{\sin \alpha}$$

$$R_a = Q\frac{\cos \alpha}{\sin \alpha} = Q \cot \alpha$$

The value of angle α is obtained as before. If we substitute the value of α in the above equation, we will get the value of R_a as before. For the lower sphere, the solution is same as above.

4. A cord *ACB* 6.1 m long, is attached at points *A* and *B* to two vertical walls, 4.88 m apart [Fig. 2.38(a)]. A pulley *C*, so small that we can neglect its radius, carries a suspended load $P = 160$ N and is free to roll without friction along the cord. Determine the position of equilibrium, as defined by the distance x, that the pulley will assume and also the tensile force in the cord.



(a)

(b)

Fig. 2.38

Solution: Neglecting friction in the pulley, we conclude that the forces exerted at *C* by the portions *AC* and *BC* of the cord must be equal. These two forces can balance the vertical force *P* only if they give a vertical resultant. This condition requires that *AC* and *BC* be equally inclined to the horizontal. From this it follows that by continuing the line BC down to point D, we obtain an isosceles triangle ACD. Thus, it is evident that ΔBFD is a 3 : 4 : 5 triangle. Now from the similarity of ΔBGH and ΔBDF we may write $GH : BH = DF : BF = 4:3$ or, using the given dimensions, $(4.88 - 2x) : 0.915 = 4 : 3$ from which = 1.83 m.

The triangle of forces [Fig. 2.38(b)] for the three forces in equilibrium at C is similar, by construction, to ΔACD. Hence, $S : P = 5 : 6$, from which we conclude that the tensile force in the cord is 133.33 N.

Alternate Solution: After drawing the free body diagram as above, using the free body diagram of point C as shown in Fig. 2.38(a) and applying Lami's theorem, we get

$$\frac{S}{\sin\left(\frac{\pi}{2}+\alpha\right)} = \frac{P}{\sin(\pi - 2\alpha)} \tag{e}$$

and $$\tan\alpha = \frac{BH}{GH} = \frac{3}{4} \tag{f}$$

Simplifying Eq. (e) for S,

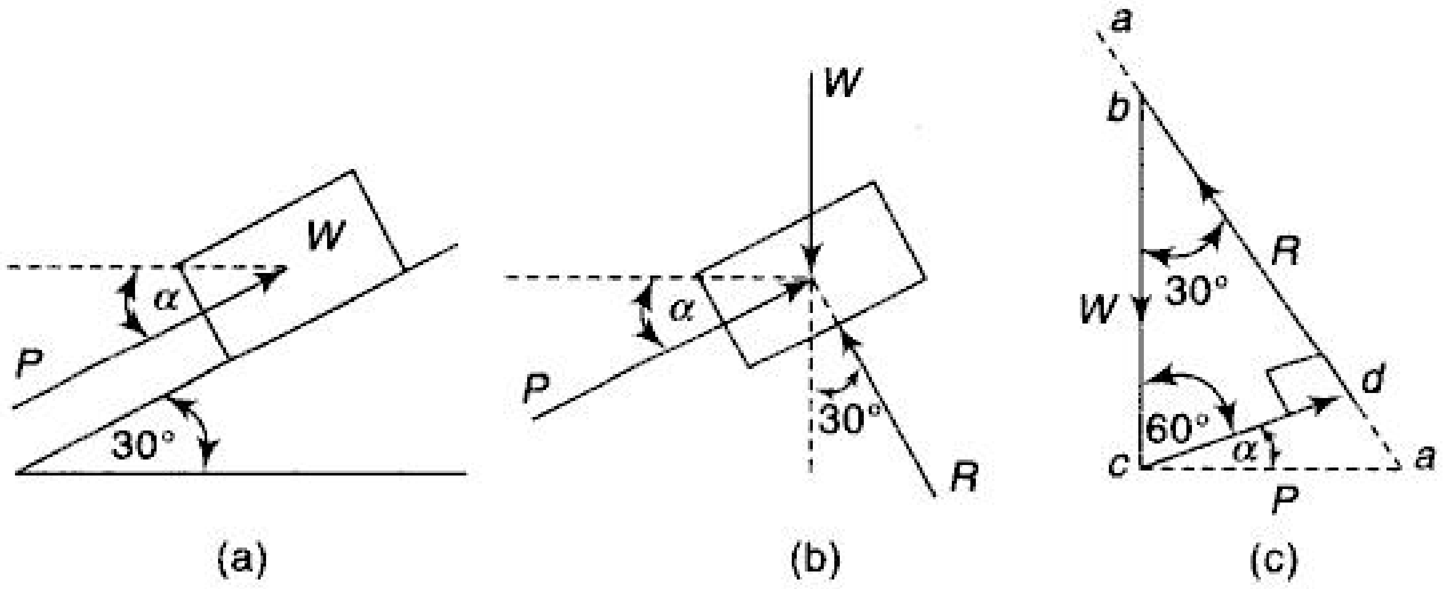
$$\Rightarrow \qquad S = P\frac{\cos\alpha}{\sin 2\alpha} = P\frac{\cos\alpha}{2\sin\alpha\cos\alpha} = \frac{P}{2\sin\alpha} \tag{g}$$

Solving for α from Eq. (f) and substituting this in Eq. (g), we obtain

$$S = 133.33 \text{ N}$$

as before.

5. Determine the magnitude and direction of the smallest force P, which will maintain the body of weight $W = 300$ N on an inclined smooth plane as shown in Fig. 2.39(a), is in equilibrium.



(a) (b) (c)

Fig. 2.39

Solution: Since the inclined plane is smooth, the pressure at the point of contact between body and inclined plane must be normal to the surface. Removing supporting inclined plane and replacing it by its reaction R, we obtain the free body diagram of the body as shown in Fig. 2.39(b).

The body is acted upon by three forces, namely, the action of gravity force W, the applied P and the reaction R. Since these three forces are in equilibrium, the vectors representing them must build a closed triangle [Fig. 2.32(c)], we begin with the known vector $\overline{bc}$ representing to a certain scale, the weight of the body, and then draw the line aa parallel to the R. The side $\overline{cd}$ will be minimum if it is perpendicular to line aa. That is P will be minimum, if it is perpendicular to aa.

From the triangle bcd, $\angle c = 90° - 30° = 60°$

$\therefore$ $\alpha = 90° - 60° = 30°$ and using the triangle bcd, we obtain

$$P = W \sin 30° = \frac{W}{2} = 150 \text{ N}$$

Alternate Solution: After drawing the free body diagram of the body of above, then applying the Lami's theorem to the free body diagram of the body as shown in Fig. 2.39(b), we get

$$\frac{W}{\sin(90° - \alpha + 30°)} = \frac{P}{\sin(\pi - 30°)} = \frac{R}{\sin(90° + \alpha)} \tag{h}$$

Using the first two of the Eq. (h), we obtain

$$\frac{W}{\cos(30° - \alpha)} = \frac{P}{\sin 30°}$$

$$P = \frac{W \sin 30°}{\cos(30° - \alpha)}$$

From Eq. (i), P will be minimum, if the denominator is maximum, i.e.,

$$\cos(30° - \alpha) = 1$$

$$\Rightarrow \quad 30° - \alpha = 0$$

$$\Rightarrow \quad \alpha = 30°$$

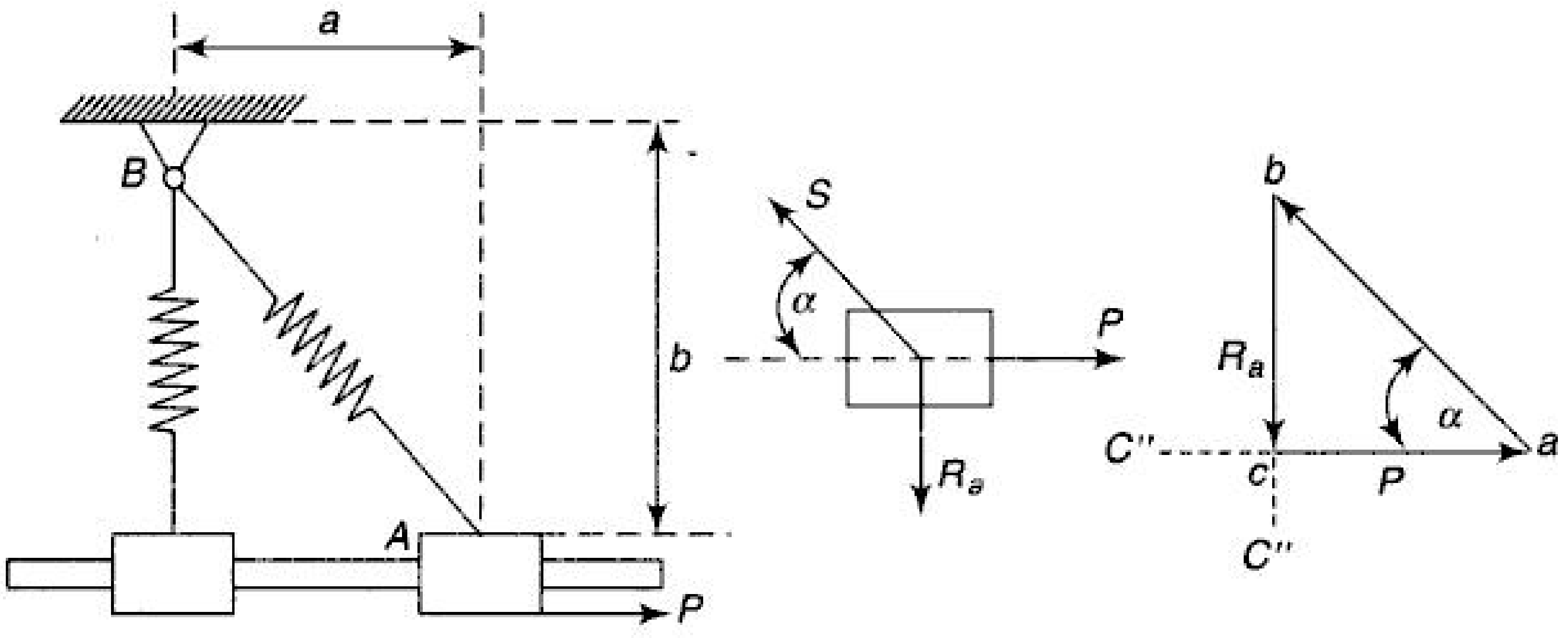
and substituting this value into Eq. (i), we get the value of

$$P = W \sin 30° = 150 \text{ N, as before.}$$

6. The sliding guide A may slide freely on horizontal frictionless rod as shown in Fig. 2.40(a). The spring attached to the sliding guide has a stiffness of 5 kN/m and is undeformed when the sliding guide is directly below the support B. Determine the magnitude of the force P required to maintain equilibrium when $a = 450$ mm and $b = 200$ mm. Also determine the reaction on the sliding guide.

   Solution: The free body diagram in which the sliding guide has been isolated from its supports and in which all forces acting upon it, both active and reactive, are indicated by vectors is shown in Fig. 2.40(b). The angle α shown in Fig. 2.40(b) is given by

$$\tan \alpha = \frac{b}{a} = \frac{0.20}{0.45}$$



(a) (b) (c)

Fig. 2.40

$$\Rightarrow \qquad \alpha = \tan^{-1}\frac{0.20}{0.45} = 23.96°$$

The spring force is given by the formula

$$S = kx$$

The deformation x is given by the equation

$$x = \sqrt{b^2 + a^2} - b = \sqrt{0.20^2 + 0.45^2} - 0.20 = 0.292 \text{ m}$$

$$S = kx = 5(\text{kN/m}) \times 0.292 \text{ (m)} = 1.46 \text{ kN}$$

The three concurrent forces P, S and R_a are a system in equilibrium and hence their free vectors must build a closed triangle. To construct this closed triangle of forces [Fig. 2.40(c)], we first lay out the vector $\overline{ab}$, representing to scale the magnitude of the spring force S. Then from ends a and b of this vector , we construct the lines ac' and bc'' parallel, respectively, to the known lines of action of the forces P and R_a. Intersection c of these two lines determines the required magnitudes of P and R_a and the arrows show the directions that these forces must have to build a closed traingle. Note that the vectors must follow one another tail to head around the triangle. The magnitudes of P and R_a may now be scaled from the drawing and the problem is sloved.

Alternately, using the trigonometric relations for a triangle, we get

$$\Rightarrow \qquad R_a = S \sin \alpha = 1.46 \sin 23.96° = 0.593 \text{ kN}$$

$$\text{and} \qquad P = S \cos \alpha = 1.46 \cos 23.96° = 1.334 \text{ kN}$$

Alternatively, from the free body diagram at point A as shown in Fig. 2.40(b) and applying Lami's theorem, we will get

$$\frac{S}{\sin \frac{\pi}{2}} = \frac{P}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{R_a}{\sin (\pi - a)}$$

$$\Rightarrow \quad R_a = S \sin \alpha$$
$$P = S \cos \alpha$$

same as before.

Note ⟫ When the angle between any two forces is $\pi/2$, then it is better to use trigonometric relations of triangle rather than Lami's theorem to obtain the forces.

7. Determine the stretch in each spring for equilibrium of the weight $W = 40$ N block as shown in Fig. 2.41(a). The springs are in equilibrium position. The stiffness of each spring is as given: $k_1 = 40$ N/m, $k_2 = 50$ N/m, and $k_3 = 60$ N/m.

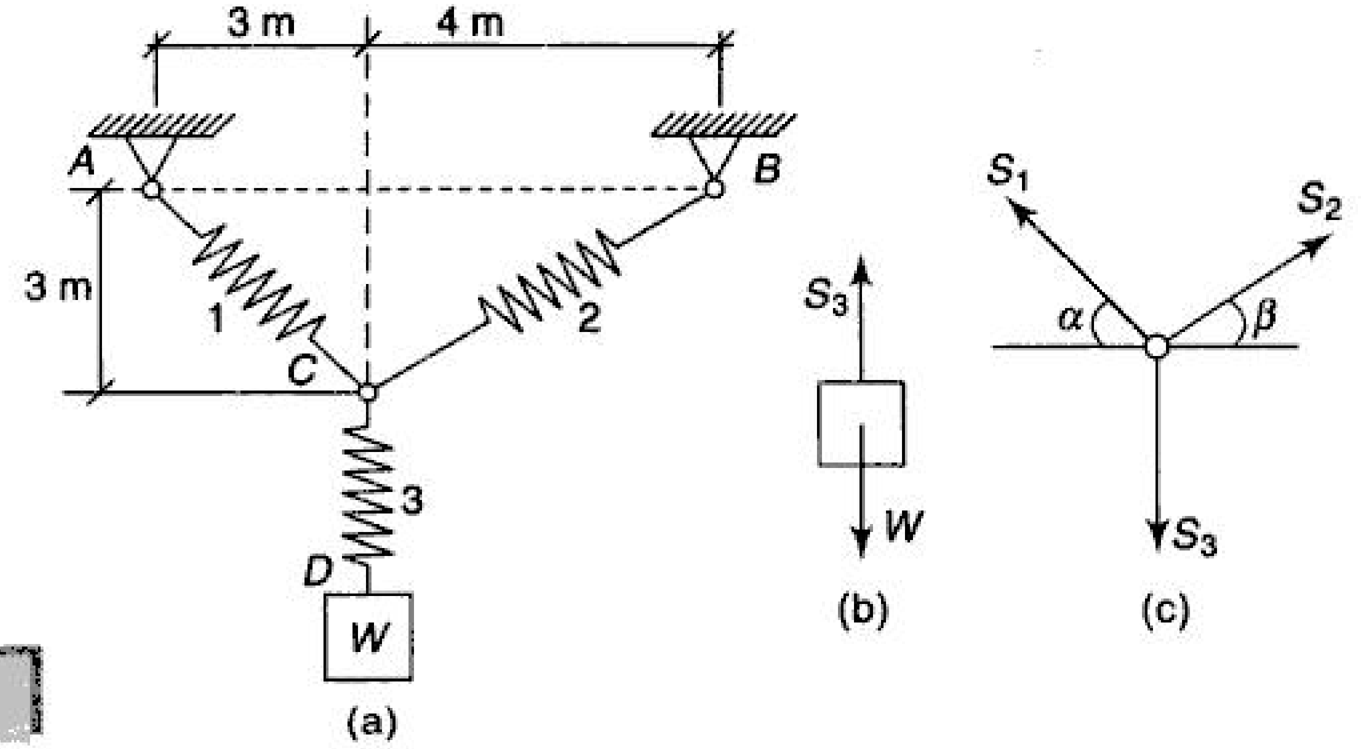


Fig. 2.41

Solution: Draw the free body diagram of the body as shown in Fig. 2.41(b). Only two forces are acting on the body, gravity force W and the reactive force caused by the spring S_3. Since the body is in equilibrium, from the law of equilibrium of two , we get the tension in the spring CD, S_3 as

$$S_3 = W$$

Now, draw the free body diagram of the point C, as shown in Fig. 2.41(c). At the joint C, three forces are acting all are reactive forces caused by the springs. The angles that springs S_1 and S_2 make with the horizontal are calculated as below:

$$\tan \alpha = \frac{3}{3} = 1 \Rightarrow \alpha = 45°$$

$$\tan \beta = \frac{3}{4} \Rightarrow \beta = 36.87°$$

Since the joint C is in equilibrium, applying Lami's theorem, we obtain

$$\frac{S_1}{\sin\left(\frac{\pi}{2} + \beta\right)} = \frac{S_2}{\sin\left(\frac{\pi}{2} + \alpha\right)} = \frac{S_3}{\sin(\pi - \alpha - \beta)}$$

Simplifying the above equation, we will get

$$\Rightarrow \qquad S_1 = \frac{S_3 \cos\beta}{\sin(\alpha+\beta)} = \frac{W\cos\beta}{\sin(\alpha+\beta)}$$

$$S_2 = \frac{S_3 \cos\alpha}{\sin(\alpha+\beta)} = \frac{W\cos\alpha}{\sin(\alpha+\beta)}$$

Substituting the values of W, α and β in the above equations, we obtain

$$\Rightarrow \qquad S_1 = \frac{W\cos\beta}{\sin(\alpha+\beta)} = \frac{40\text{N}\times\cos 36.87^\circ}{\sin(45^\circ+36.87^\circ)} = 32.33\text{ N}$$

$$S_2 = \frac{W\cos\alpha}{\sin(\alpha+\beta)} = \frac{40\text{N}\times\cos 45^\circ}{\sin(45^\circ+36.87^\circ)} = 28.57\text{ N}$$

The spring force is given by the formula

$$S = kx$$

$$\Rightarrow \qquad S_1 = k_1x_1 \qquad \text{(k)}$$

$$S_2 = k_2x_2$$

$$S_3 = k_3x_3$$

Substituting the values of S_1, S_2, S_3, k_1, k_2 and k_3, in the above equations, we get the stretch in each spring as below:

$$x_1 = \frac{S_1}{k_1} = \frac{32.33}{40} = 0.81\text{ m}$$

$$x_2 = \frac{S_2}{k_2} = \frac{28.57}{50} = 0.57\text{ m}$$

$$x_3 = \frac{S_3}{k_3} = \frac{40}{60} = 0.67\text{ m}$$

8. Determine the axial forces S_1 and S_2 induced in the bars AC and BC in Fig. 2.42(a) due to the action of the horizontal force P applied load at C. The bars are hinged together at C and to the foundation at A and B.

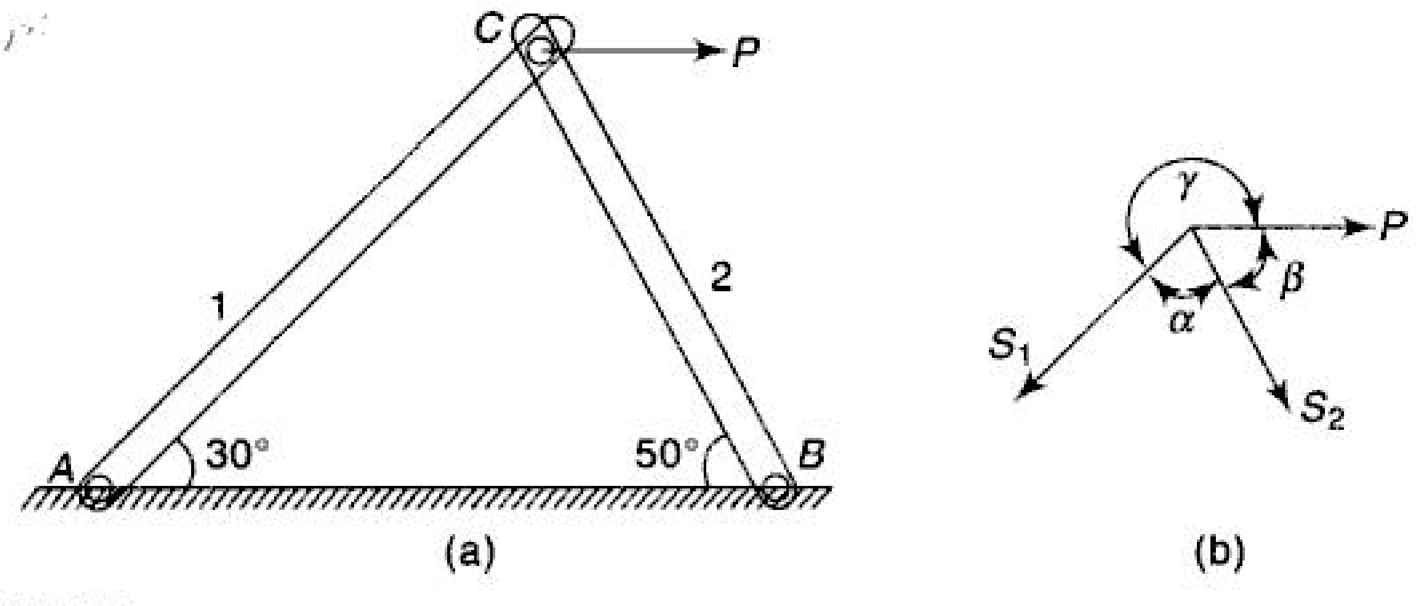


Fig. 2.42

Solution: The free body diagram of the point C is shown in Fig. 2.42(b). Here we assumed that both the bars are in tension. Since the point C is in equilibrium, applying Lami's theorem, we get

$$\frac{P}{\sin\alpha} = \frac{S_1}{\sin\beta} = \frac{S_2}{\sin\gamma} \tag{l}$$

We have $\alpha = 100°$, $\beta = 50°$ and $\gamma = 210°$; substituting these values into the above equation (l), we get

$$\frac{S_1}{\sin\beta} = \frac{P}{\sin\alpha}$$

$$\Rightarrow \qquad S_1 = \frac{P\sin\beta}{\sin\alpha} = \frac{P\sin 50°}{\sin 100°} = 0.78\,P$$

$$\frac{S_2}{\sin\gamma} = \frac{P}{\sin\alpha}$$

$$\Rightarrow \qquad S_2 = \frac{P\sin\gamma}{\sin\alpha} = \frac{P\sin 210°}{\sin 100°} = -0.508\,P$$

From the result, the assumed direction of tensile force S_1 is positive, i.e., assumed direction is correct and assumed direction of the force S_2 is wrong, as the result is negative. So, the force S_2 is compressive, i.e., opposite to the assumed direction.

9. A ball of weight W rests upon a smooth horizontal plane and has attached to its center two strings AB and AC which pass over frictionless pulleys at B and C and carry loads P and Q, respectively, as shown in Fig. 2.43(a). If the string AB is horizontal, find the angle α that the string AC makes with the horizontal when the ball is in a position of equilibrium. Also find the pressure R between the ball and the plane.

Solution: Draw the free body diagrams of weights as shown in Fig. 2.43(b) and *c* discussed as before. From the law of equilibrium of collinear forces, we get

$$S_1 = P \text{ and } S_2 = Q$$

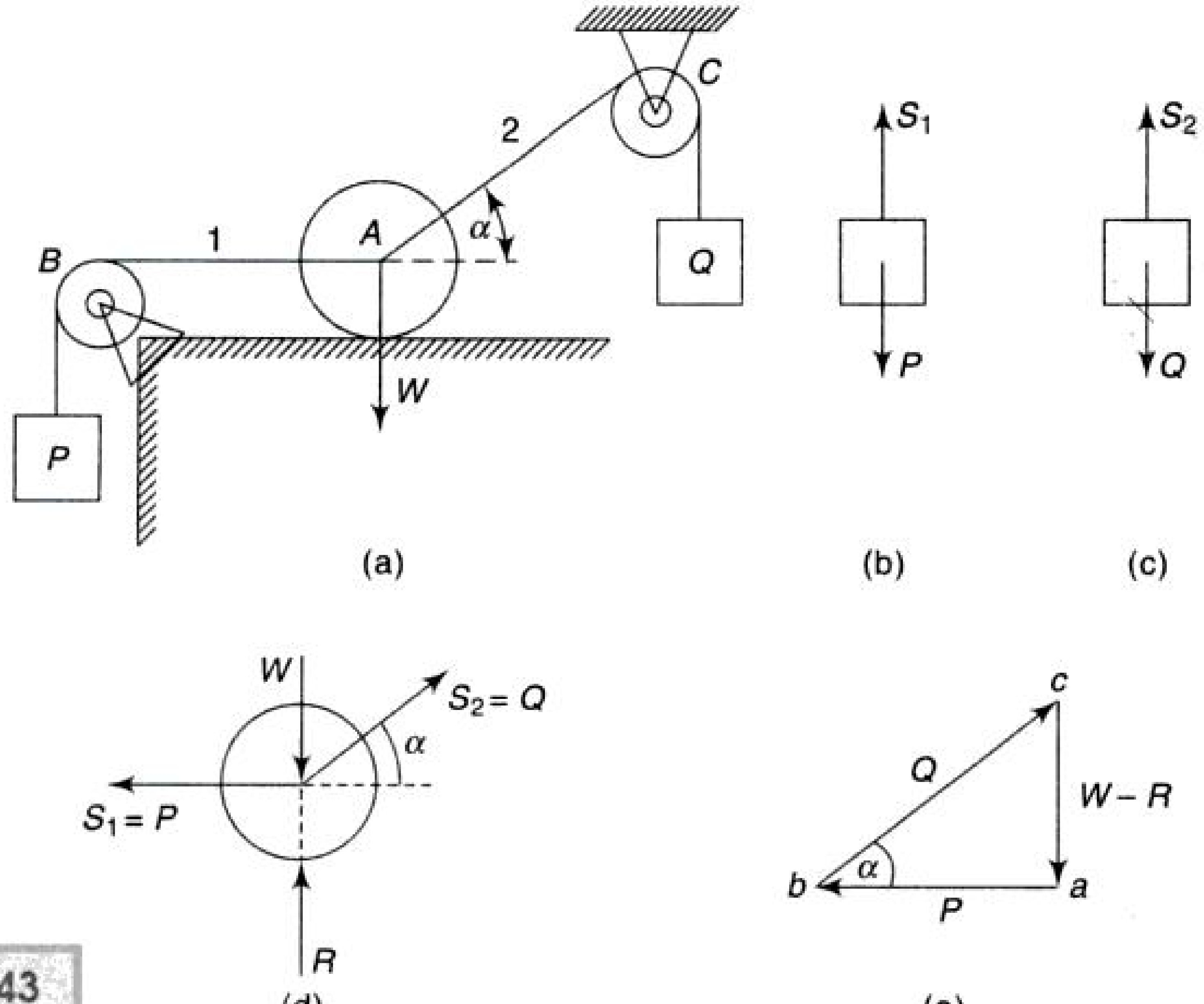
Draw the free body diagram of the ball as shown in Fig. 2.43(d) as discussed before. Since W and R are collinear forces, replace them with the resultant, which is equal to $W - R$. Draw the triangle of forces as shown in Fig. 2.43(e). Using the trigonometric relations, we get

$$\cos\alpha = \frac{P}{Q}$$

$$W - R = \sqrt{Q^2 - P^2} \tag{m}$$

$$\Rightarrow \qquad R = W - \sqrt{Q^2 - P^2}$$

The angle α that the string AC makes with the horizontal when the ball is in a position of equilibrium and also the pressure R between the ball and the plane is given by equation (m).



(a) (b) (c)

Fig. 2.43 (d) (e)

Important Terms and Concepts

Concurrent forces in a plane
Lami's theorem
Statically indeterminate
Equilibrium of concurrent forces in a plane
Equilibrant
Redundant constraints
Statically determinate

SUMMARY

- If all the forces in a system lie in a single plane and pass through a single point, then the system constitutes a coplanar concurrent force system.
- Equilibrium of concurrent forces in a plane: The resultant of any number of concurrent forces in a plane is given by the closing side of the polygon of forces obtained by successive geometric addition of their free vectors. In the particular case where the end of the last vector coincides with the beginning of the first, the resultant vanishes and the system is in equilibrium.
- The condition of equilibrium for any system of concurrent forces in a plane: If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces, or rather their free vectors, when geometrically added must form a closed polygon. It should be noted that in any closed polygon of forces, the vectors must follow one another tail to head around the polygon.
- The closed polygon of forces is constructed to scale and the magnitudes of the reactions measured directly from the drawing, is called a graphical solution of the problem.
- The closed polygon of forces is constructed not to scale and the magnitudes of the reactions computed from the trigonometry, is called a trigonometrical solution of the problem.

- Lami's Theorem: If three concurrent forces are acting on a body, keep it in equilibrium, then each force is proportional to the sine of the angle between the other two forces and the constant of proportionality is the same.
- In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we can determine definitely the magnitudes of two reactive forces. Then the problem is known as statically determinate.
- In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we cannot determine definitely the magnitudes of more than two reactive forces. If more than two reactive forces exist in the given problem, then there is no way to determine their magnitudes definitely and the problem is said to be statically indeterminate.
- A force, which is equal, opposite and collinear to the resultant of a concurrent force system is known as the equilibrant of the concurrent force system. Equilibrant, is the force which, when applied to the body acted upon by the concurrent force system, keeps the body in equilibrium.
- Redundant constraint: Supports in excess of the necessary and sufficient to completely constrain the body in the plane of the figure are called redundant constraints.

Important Formulae

- When a body is subjected to only three forces and is in equilibrium, they must form a closed triangle when we apply the polygon law of forces. Applying the sine rule for the force triangle, we have the relations

$$\sin \beta = (Q/R) \sin \alpha \qquad \sin \gamma = (P/R) \sin \alpha$$

In ΔABC, $AB = P$; $BC = Q$; $CA = R$;

$\pi - \alpha$ = angle between forces P and Q
β = angle between forces P and R
γ = angle between forces R and Q

- Lami's theorem: $\dfrac{P}{\sin \alpha} = \dfrac{Q}{\sin \beta} = \dfrac{R}{\sin \gamma} = k$

PRACTICE SET 2.3

Review Questions

1. Write the statement which represents the condition of equilibrium for any system of concurrent forces in a plane.
2. State and prove Lami's theorem.
3. Define equilibrant.
4. What is meant by statical indeterminacy?
5. Explain the term redundant constraints.
6. Differentiate between resultant and equilibrant.

Objective Questions

1. When more than three concurrent forces are in equilibrium, select the condition that is satisfied.
   (a) All the forces must have equal magnitude.
   (b) Polygon representing the forces will not close.
   (c) The last side of the polygon will represent the resultant.
   (d) Polygon representing the forces will close. [*Ans.* (d)]

2. The force that cancels the effect of the force system acting on the body is known as
   (a) Resultant (b) Neutral force (c) Balancing force (d) Equilibrant
   [*Ans.* (d)]
3. Consider the following statements :
   I. The resultant of any number of concurrent forces in a plane is given by the closing side of the polygon of forces obtained by successive geometric addition of their free vectors.
   II. If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces, or rather their free vectors, when geometrically added must form a closed polygon.

   Of these statements
   (a) 1alone is correct (b) 2 alone is correct
   (c) 1 and 2 are correct (d) Neither 1 nor 2 is correct [*Ans.* (c)]
4. Match the list I with the list II using the codes given below:

I	II
A. Equilibrium	1. A force, which is equal, opposite and collinear to the resultant of a concurrent force system.
B. Equilibrant	2. The resultant vanishes.
C. Statically indeterminate	3. Supports in excess of those necessary and sufficient to completely constrain the body in the plane of the figure.
D. Redundant constraints	4. In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane contains more than two reactive forces.

Codes:

(a)	A-1	B-2	C-3	D-4	
(b)	A-2	B-3	C-4	D-1	
(c)	A-2	B-1	C-4	D-3	
(d)	A-4	B-3	C-2	D-1	[*Ans.* (c)]

5. Assertion (A): In dealing with the equilibrium of constrained bodies under the action of concurrent forces in one plane, we cannot determine definitely the magnitudes of more than two reactive forces and the problem is said to be statically indeterminate.

   Reason (R): The resolution of a given force into more than two coplanar concurrent components is an indeterminate problem.

   Select the answer from the following codes:
   (a) A and R are true and R is the correct explanation of A.
   (b) A and R are true and R is not the correct explanation of A.
   (c) A is true and R is false.
   (d) Both A and R are false. [*Ans.* (a)]

PROBLEM SET 2.3

1. An electric-light fixture of weight $Q = 178$ N is supported as shown in Fig. A. Determine the tensile forces S_1 and S_2 in the wires *BA* and *BC* if their angles of inclination are as shown. (*Ans.* $S_1 = 130.3$ N; $S_2 = 92.14$ N)
2. A ball of weight $Q = 53.4$ N rests in a right-angled trough, as shown in Fig. B. Determine the forces exerted on the sides of the trough at *D* and *E* if all surfaces are perfectly smooth. (*Ans.* $R_d = 46.25$ N; $R_e = 26.7$ N)

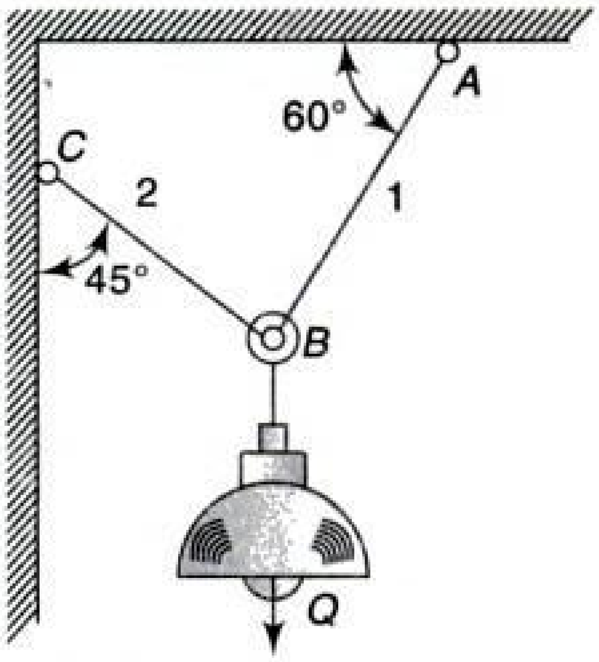


Fig. A

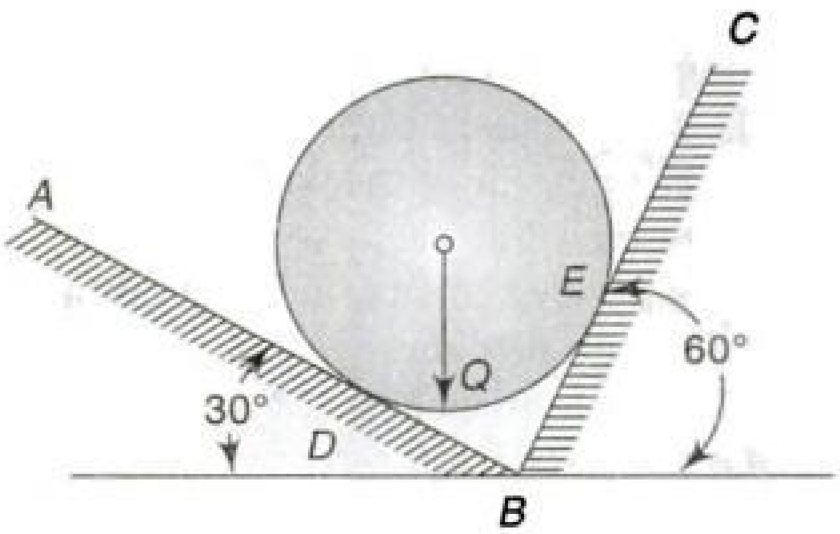


Fig. B

3. A ball rests in a trough as shown in Fig. C. Determine the angle of tilt θ with the horizontal so that the reactive force at B will be one-third at A if all surfaces are perfectly smooth. (*Ans.* $\theta = 49.11°$)

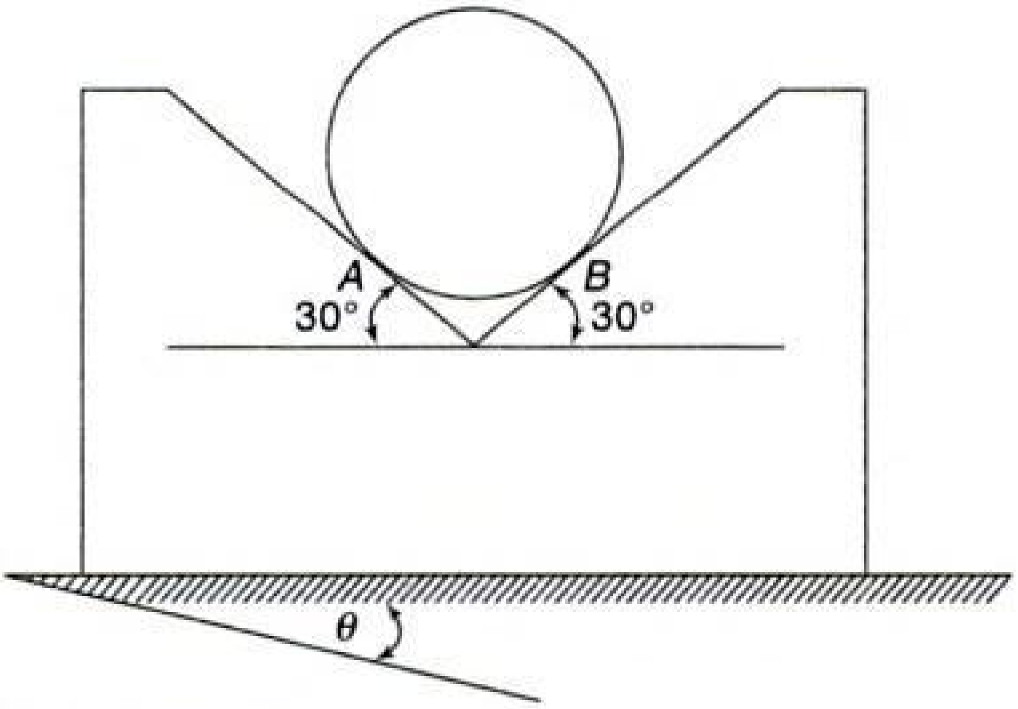


Fig. C

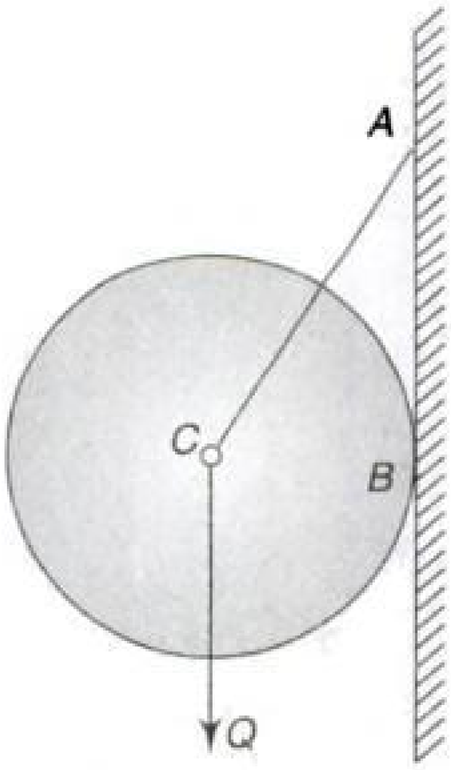


Fig. D

4. A circular roller of weight $Q = 445$ N and radius $r = 152$ mm hangs by a tie rod AC = 304 mm and rests against a smooth vertical wall at B, as shown in Fig. D. Determine the tension S in the tie rod and the force R_b exerted against the wall at B. (*Ans.* $S = 513.84$ N; $R_b = 256.92$ N)
5. What axial forces does the vertical load P induce in the members of the system shown in Fig. *E*? Neglect the weights of the members themselves and assume an ideal hinge at A and a perfectly flexible string BC. (*Ans.* $S_1 = P \sec \alpha$, tension; $S_2 = P \cos \alpha$, compression)
6. What axial forces does the vertical load P induce in the members of the system shown in Fig. F? Make the same idealizing assumptions as in Prob. 5. (*Ans.* $S_1 = P \tan \alpha$, tension; $S_2 = P \sec \alpha$, compression)
7. A right circular roller of weight W rests on a smooth horizontal plane and is held in position by an inclined bar AC as shown in Fig. G. Find the tension S in the bar AC and the vertical reaction R_b at B if there is also a horizontal force P acting at. (*Ans.* $S = P \sec \alpha$; $R_b = W + P \tan \alpha$)

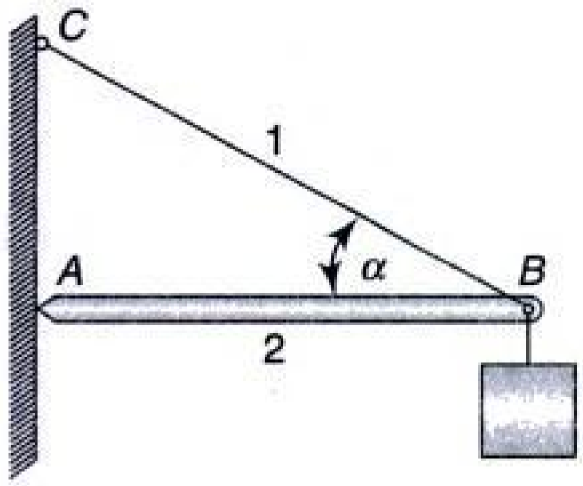


Fig. E

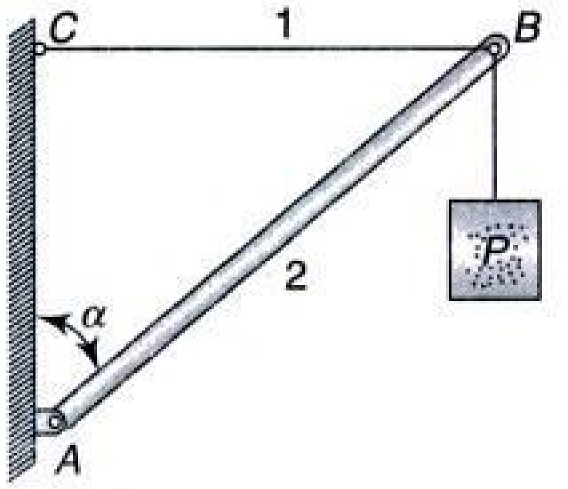


Fig. F

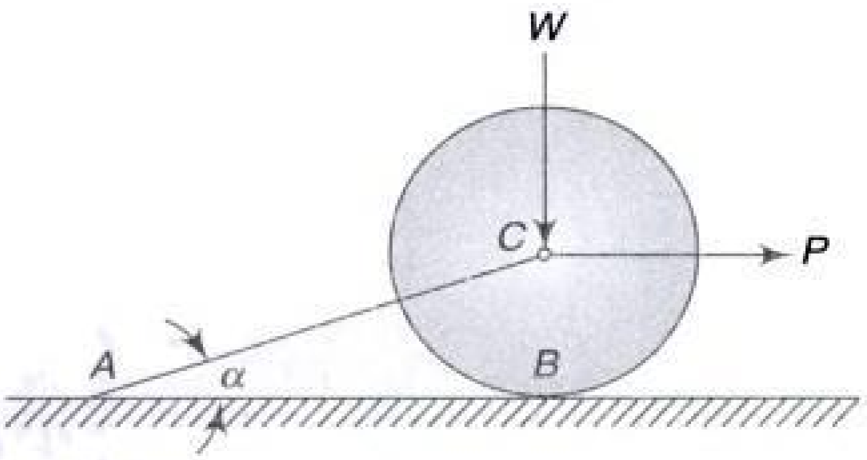


Fig. G

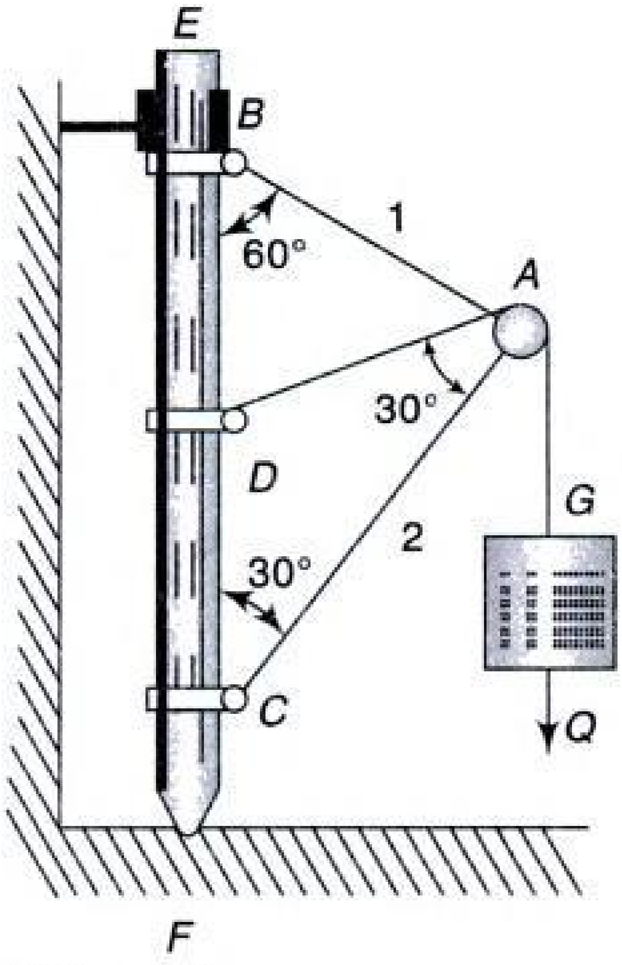


Fig. H

8. A pulley A is supported by two bars AB and AC which are hinged at points B and C to a vertical mast EF (Fig. H). Over the pulley hangs a flexible cable DG which is fastened to the mast at D and carries at the other end G a load $Q = 20$ kN. Neglecting friction in the pulley, determine the forces produced in the bars AB and AC. The angles between the various members are shown in the figure.

(*Ans.* $S_2 = 34.64$ kN; $S_1 = 0$)

9. Two smooth circular cylinders, each of weight $W = 445$ N and radius $r = 152$ mm, are connected at their centers by a string AB of length $l = 406$ mm and rest upon a horizontal plane, supporting above them a third cylinder of weight $Q = 890$ N and radius $r = 152$ mm (Fig. I). Find the forces S in the string and the pressures produced on the floor at the points of contact D and E.

(*Ans.* $S = 398$ N, tension; $R_d = R_e = 890$ N)

10. Two identical rollers, each of weight $Q = 445$ N, are supported by an inclined plane and a vertical wall as shown in Fig. J. Assuming smooth surfaces, find the reactions induced at the points of support A, B and C.

(*Ans.* $R_a = 385.4$ N; $R_b = 642.3$ N; $R_c = 513.84$ N)

11. A weight Q is suspended from point B of a cord ABC, the ends of which are pulled by equal weights P overhanging small pulleys A and C which are on the same level

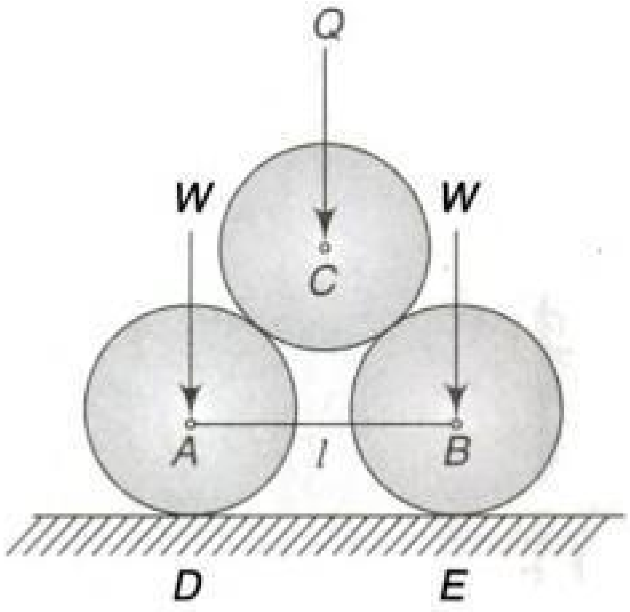


Fig. I

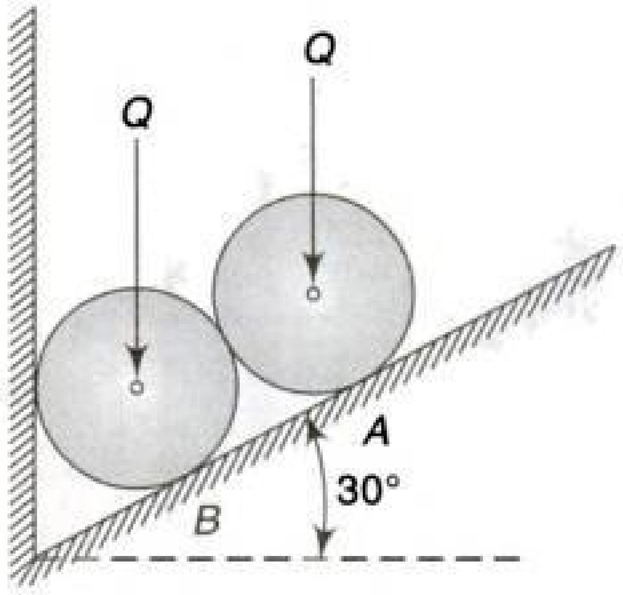


Fig. J

(Fig. K). Neglecting the radii of the pulleys, determine the sag BD if $l = 3.66$ m, $P = 89$ N, and $Q = 44.5$ N. (*Ans.* $BD = 0.473$ m)

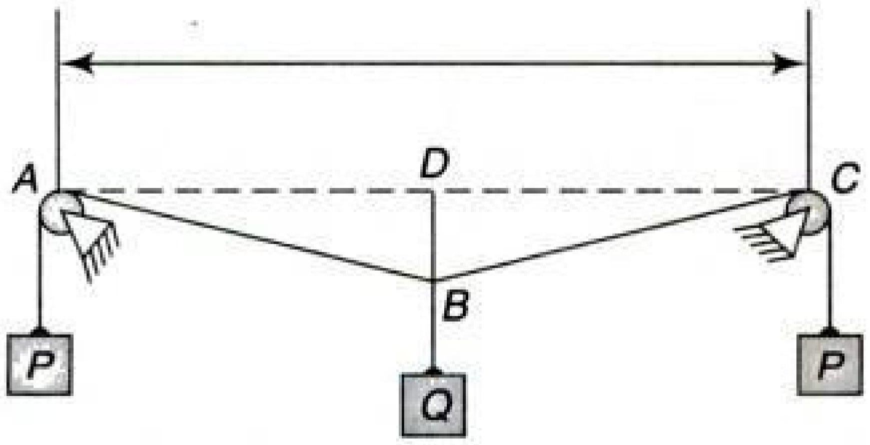


Fig. K

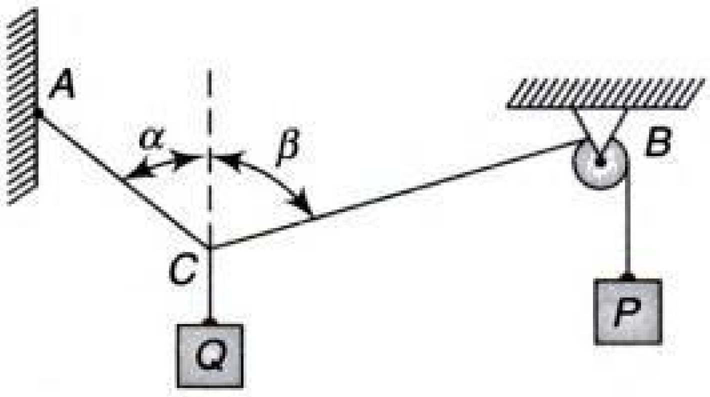


Fig. L

12. A weight Q is suspended from a small ring C, supported by two cords AC and BC (Fig. L). The cord AC is fastened at A while the cord BC passes over a frictionless pulley at B and carries the weight P as shown. If $P = Q$ and $\alpha = 50°$, find the value of the angle β. (*Ans.* $\beta = 80°$)
13. In Fig. M, weights P and Q are suspended in a vertical plane by strings 1, 2, 3 arranged as shown in Fig. M.

    Find the tension induced in each string if $P = 30$ kN, $Q = 40$ kN, $\alpha = 40°$ and $\beta = 50°$. Also find the inclination γ of segment CD to the vertical.

    (*Ans.* $S_1 = 132.35$ kN, $S_2 = 111.05$ kN, $S_3 = 89.59$ kN, $\gamma = 110°$)
14. Three equal inextensible strings of negligible weight are knotted together to form an equilateral triangle ABC and a weight W is suspended from A. If the triangle and weight to be supported with BC horizontal by means of two strings at B and C as shown in Fig. N, each at an angle of $\alpha = 135°$ with BC, find the tension in the string 3. (*Ans.* $S_3 = 0.211$ W)
15. A force P is applied at point C as shown in Fig. O. Determine the value of angle α for which the larger of the string tension is as small as possible and the corresponding values of tension in the strings 1 and 2. (*Ans.* $\alpha = 60°$, $S_1 = S_2 = 0.577$ P)
16. A ball of weight W is suspended from a string of length l and is pulled by a horizontal force Q. The weight is displaced by a distance d from the vertical position as shown in Fig. P. Find the angle α, force Q required and the tension S in the string in the displaced position, if the ball is in equilibrium.

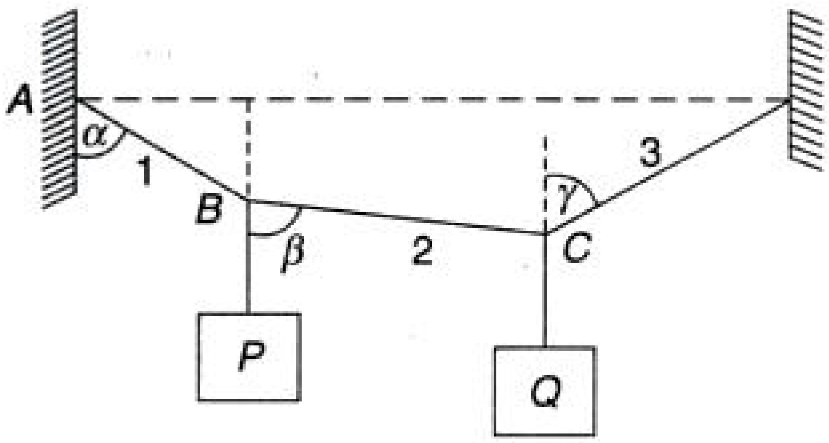


Fig. M

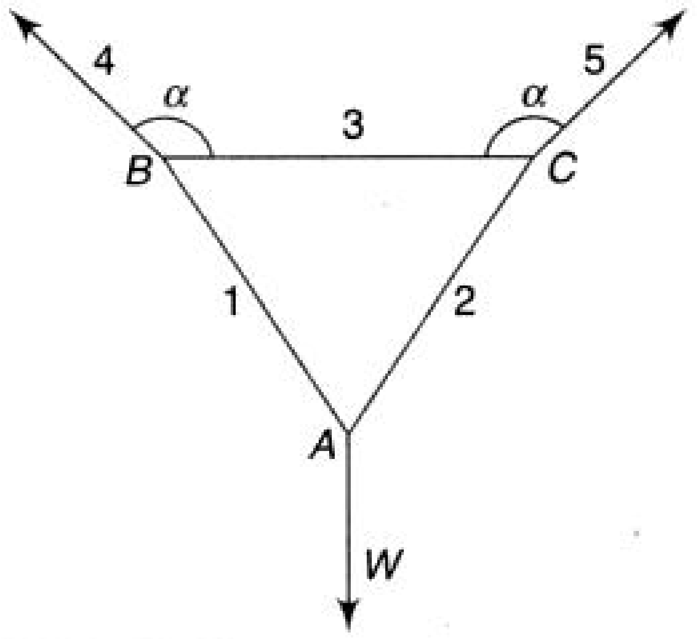


Fig. N

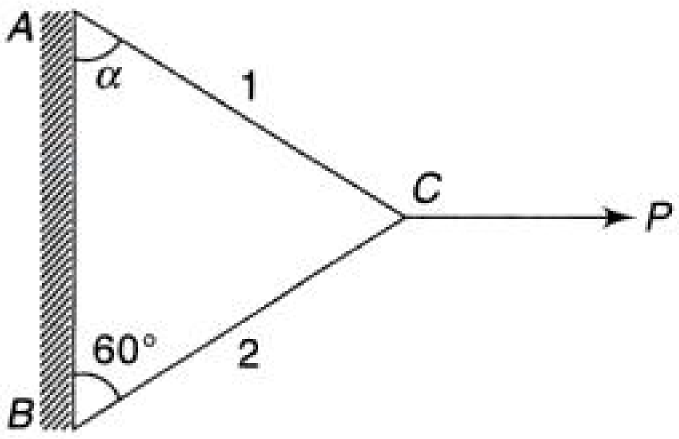


Fig. O

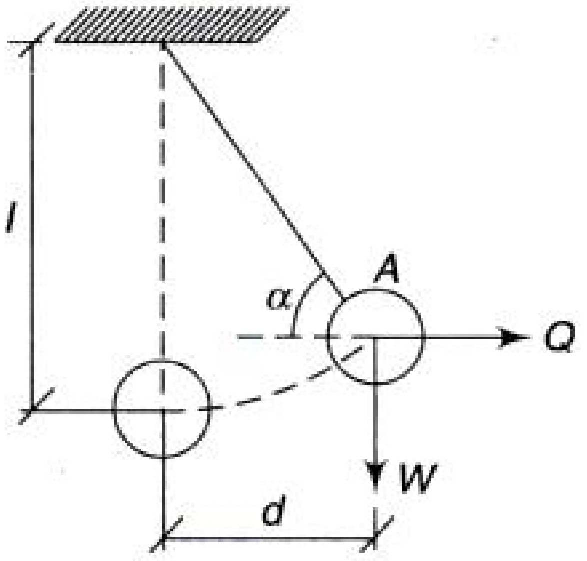


Fig. P

$$\left(Ans.\ \alpha = \cos^{-1}\frac{d}{l}, Q = \frac{Wd}{\sqrt{l^2 - d^2}}, S = \frac{Wl}{\sqrt{l^2 - d^2}} \right)$$

17. A weight 100 N hangs by an inextensible string from a fixed point A. The string is drawn out of the vertical by applying a force 50 N to the weight at point B as shown in Fig. Q. In what direction must this force be applied in order that, in equilibrium, the direction of the string from the vertical may have its greatest value. What is the amount of greatest deflection. Find also the tension in the string. (*Ans.* $\alpha = 90°$, $\beta = 30°$, $S = 86.6$ N)
18. Three bars in one plane, hinged at their ends as shown in Fig. R, are submitted to the action of a force $P = 44.5$ N applied at the hinge as shown. Determine the magnitude of the force that it will be necessary to apply at the hinge in order to keep the system of bars in equilibrium if the angles between the bars and the lines of action of the forces are as given in the figure. (*Ans.* $Q = 72.54$ N)

*19. A rigid bar with rollers of weights $P = 222.5$ N and $Q = 445$ N at its ends is supported inside a circular ring in a vertical plane as shown in Fig. S. The radius of the ring and the length AB are such that the radii AC and BC form a right angle at C; that is, $\alpha + \beta = 90°$. Neglecting friction and the weight of the bar AB, find the configuration of equilibrium as defined by the angle $(\alpha - \beta)/2$ that makes with the horizontal. Find also the reactions R_a and R_b and the compressive force S in the bar AB.
(*Ans.* $(\alpha - \beta)/2 = 18°26'$; $R_a = 298.5$ N; $R_b = 597$ N; $S = 281.5$ N)

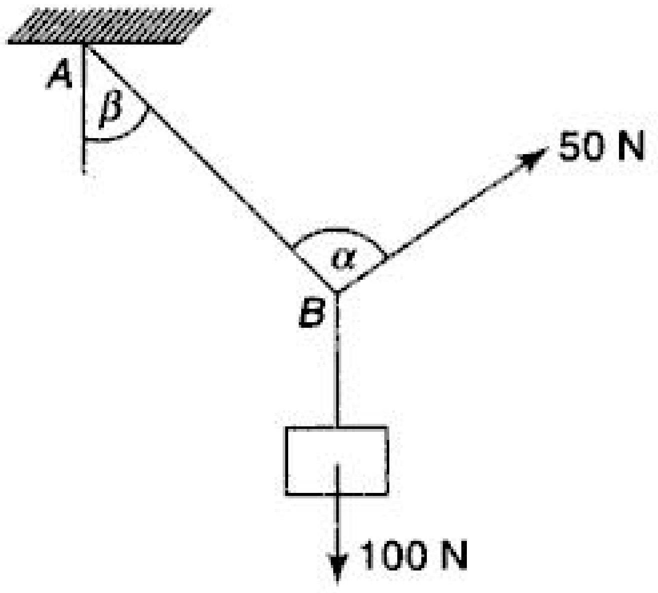


Fig. Q

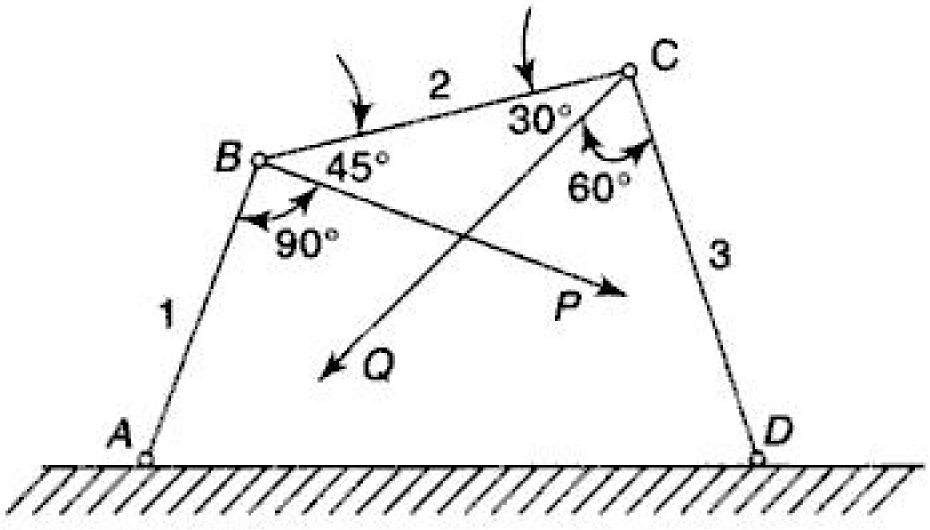


Fig. R

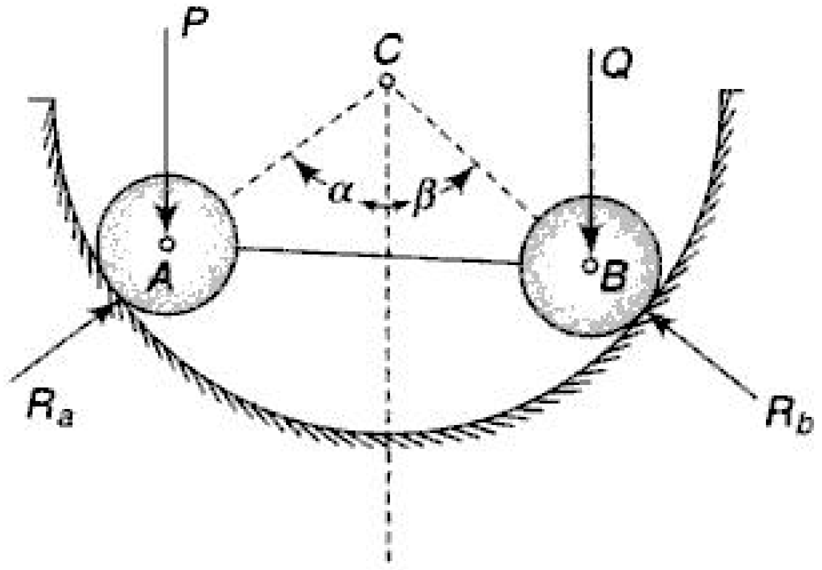


Fig. S

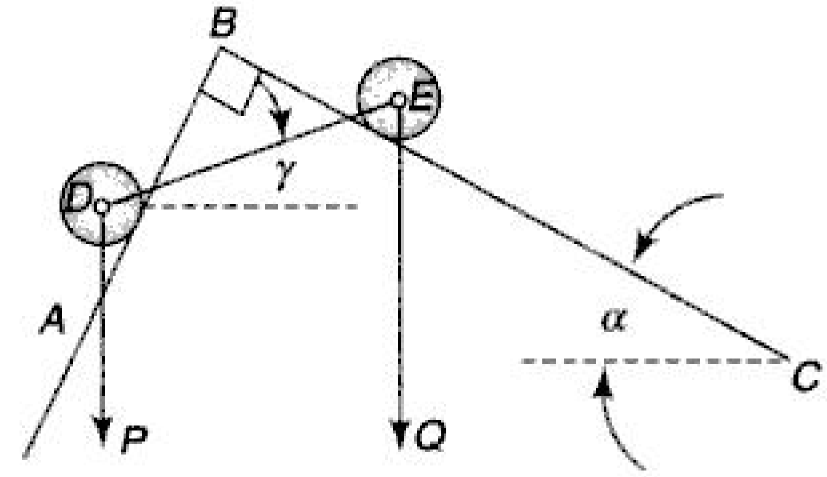


Fig. T

*20. Two rollers of weights P and Q are connected by a flexible string DE and rest on two mutually perpendicular planes AB and BC, as shown in Fig. T. Find graphically the tension S in the string and the angle φ that it makes with the horizontal when the system is in equilibrium. The following numerical data are given: $P = 267$ N, $Q = 445$ N, $\alpha = 30°$. Assume that the string is inextensible and passes freely through slots in the smooth inclined planes AB and BC. (*Ans.* $S = 321$N ; $\varphi = 16°$)

2.4 METHOD OF PROJECTIONS

Previously, we have handled all problems of composition, resolution and equilibrium of concurrent forces in a plane by using the method of geometric addition of their free vectors. These same problems can also be solved by a method of algebraic addition of the *projections* of the given forces on rectangular coordinate axes x and y taken in the plane of action of the forces. To develop this method, let us consider first the case of two forces F_1 and F_2, applied at point A (Fig. 2.44) and making with the positive directions of the coordinate axes the angles α_1, β_1 and α_2, β_2, respectively. Their resultant R is obtained from the parallelogram of forces, and the angles that it makes with the x and y axes, respectively, will be denoted by α and β. Considering, now, all forces projected onto the x axis, we find for these projections the values $F_1 \cos \alpha_1$, $F_2 \cos \alpha_2$, and $R \cos \alpha$, and we see that

$$R \cos \alpha = F_1 \cos \alpha_1 + F_2 \cos \alpha_1 \qquad \text{(a)}$$

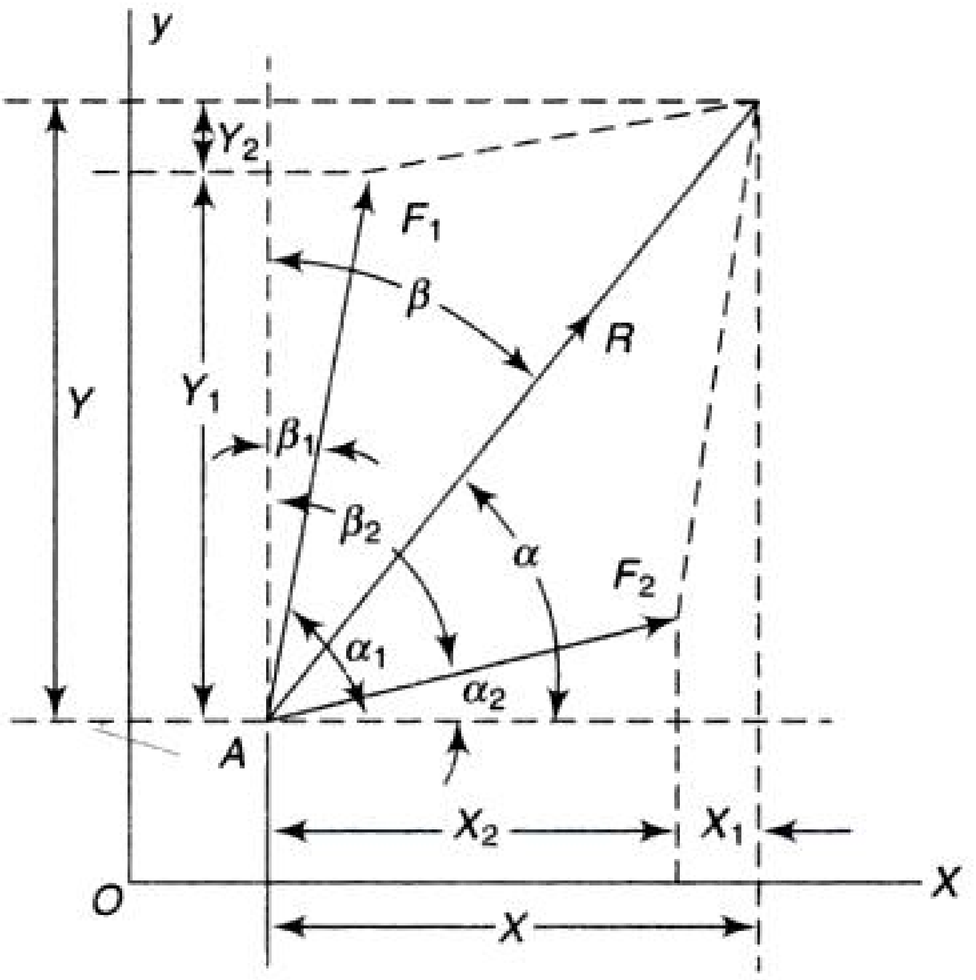


Fig. 2.44

In the same manner, considering all forces projected onto the y-axis, we obtain

$$R \cos \beta = F_1 \cos \beta_1 + F_2 \cos \beta_2 \tag{b}$$

Thus from Eqs (a) and (b) it may be stated that the *projection of the resultant of two forces on any axis is equal to the algebraic sum of the projections of its components on the same axis.*

By successive applications of the principle of the parallelogram of forces, the above conclusion can be obtained for any number of concurrent forces F_1, F_2, ..., F_n in a plane. Using, for the projections of the various forces, the following notations

$$X_i = F_i \cos \alpha_i \qquad Y_i = F_i \cos \beta_i$$
$$X = R \cos \alpha \qquad Y = R \cos \beta$$

We obtain

$$X = X_1 + X_2 + \dots + X_n = \Sigma X_i \tag{1}$$
$$Y = Y_1 + Y_2 + \dots + Y_n = \Sigma Y_i$$

where the summations are understood to include all forces in the system. *Thus, the projections, on the coordinate axes, of the resultant of a system of concurrent forces F_1, F_2, . . . , F_n acting in one plane are equal to the algebraic sum of the corresponding projections of the components.*

Knowing the magnitudes and directions of the various forces, their projections X_1, X_2, . . . , X_n and Y_1, Y_2, . . . , Y_n on the rectangular coordinate axes x and y, respectively, may be computed and tabulated in systematic order. The algebraic summations, indicated by Eq. (c), for determining the projections X and Y of the resultant may then be made and the magnitude and direction of the resultant computed from the following equations:

$$R = \sqrt{X^2 + Y^2}, \qquad \tan \alpha = \frac{Y}{X} \tag{2}$$

The use of Eq. (2) for determining the resultant of a given system of concurrent forces in a plane is sometimes more advantageous than the method of geometric addition of the vectors representing these forces as discussed in Section 2.2.

$$\alpha = \text{arc tan } \frac{Y}{X} = \text{arc tan } \frac{+265.8}{-361.6} = 143°41'$$

We see from the signs of X and Y that the resultant lies in the second quadrant; hence the angle α is measured out in the counterclockwise direction from the positive end of the x axis, as shown.

2. A load $P = 4450$ N is bracketed from a vertical wall by two bars AB and AC hinged together at A and to the wall at B and C as shown in Fig. 2.46. Using the method of projections, compute the axial forces S_1 and S_2 induced in these bars.

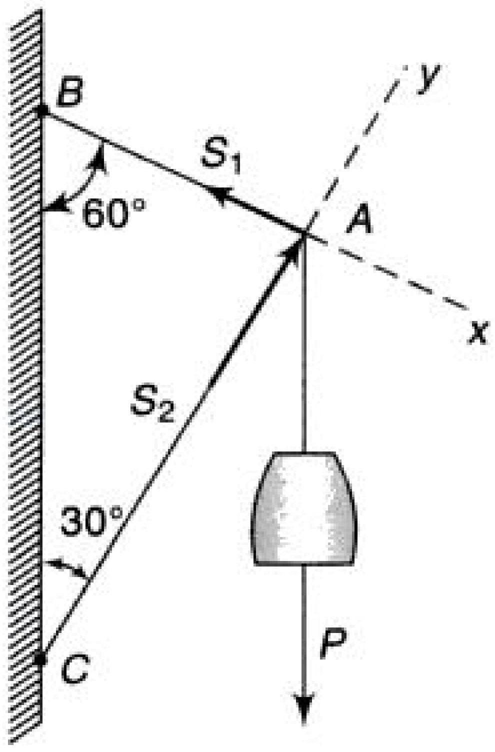


Fig. 2.46

Solution: We first make a free body of the pin A, replacing the bars AB and AC by the reactions S_1 and S_2 directed as shown in the figure. Then choosing coordinate axes x and y as shown, the equations of equilibrium (3) become

$$-S_1 + 0.500\,P = 0$$
$$+ S_2 - 0.866\,P = 0$$

giving $S_1 = 0.500P = 2250$ N tension and

$S_1 = 0.866P = 3853.\ 7$ N compression

3. A small ring B carries a vertical load P and is supported by two strings BA and BC, the latter of which carries at its free end a weight $Q = 44.5$ N, as shown in Fig. 2.47. Find the magnitude of the load P and the tension S in the string AB if the angles that the strings AB and BC make with the vertical are as shown in the figure and the system is in equilibrium.

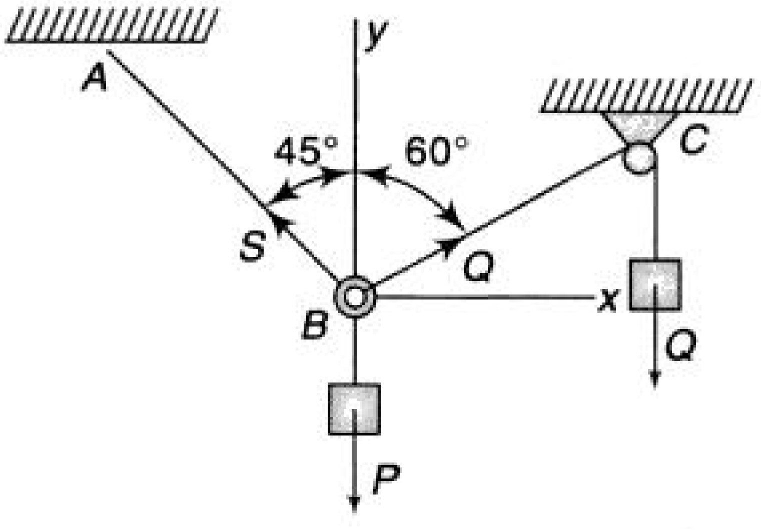


Fig. 2.47

Solution: Neglecting friction in the pulley at C, it is evident that the tension in the string BC is equal to the load Q. Thus, acting on the ring B, we have three concurrent forces in a plane that are in equilibrium. Taking coordinate axes x and y as shown, Eq. (3) become

$$Q \cos 30° - S \cos 45° = 0$$
$$Q \cos 60° + S \cos 45° = P$$

From the first of these equations, we find $S = 22.25\ \sqrt{6} = 54.50$ N. Then substituting in the second equation, we obtain

$$P = 22.25\ (1 + \sqrt{3}) = 60.8\text{N}$$

4. Three bars in one plane, hinged at their ends as shown in Fig. 2.48(a), are submitted to the action of a force $P = 44.5$ N applied at the hinge B as shown. Determine the magnitude of the force Q that it will be necessary to apply at the hinge C in order to keep the system of bars in equilibrium if the angles between the bars and the lines of action of the forces are as given in the figure.

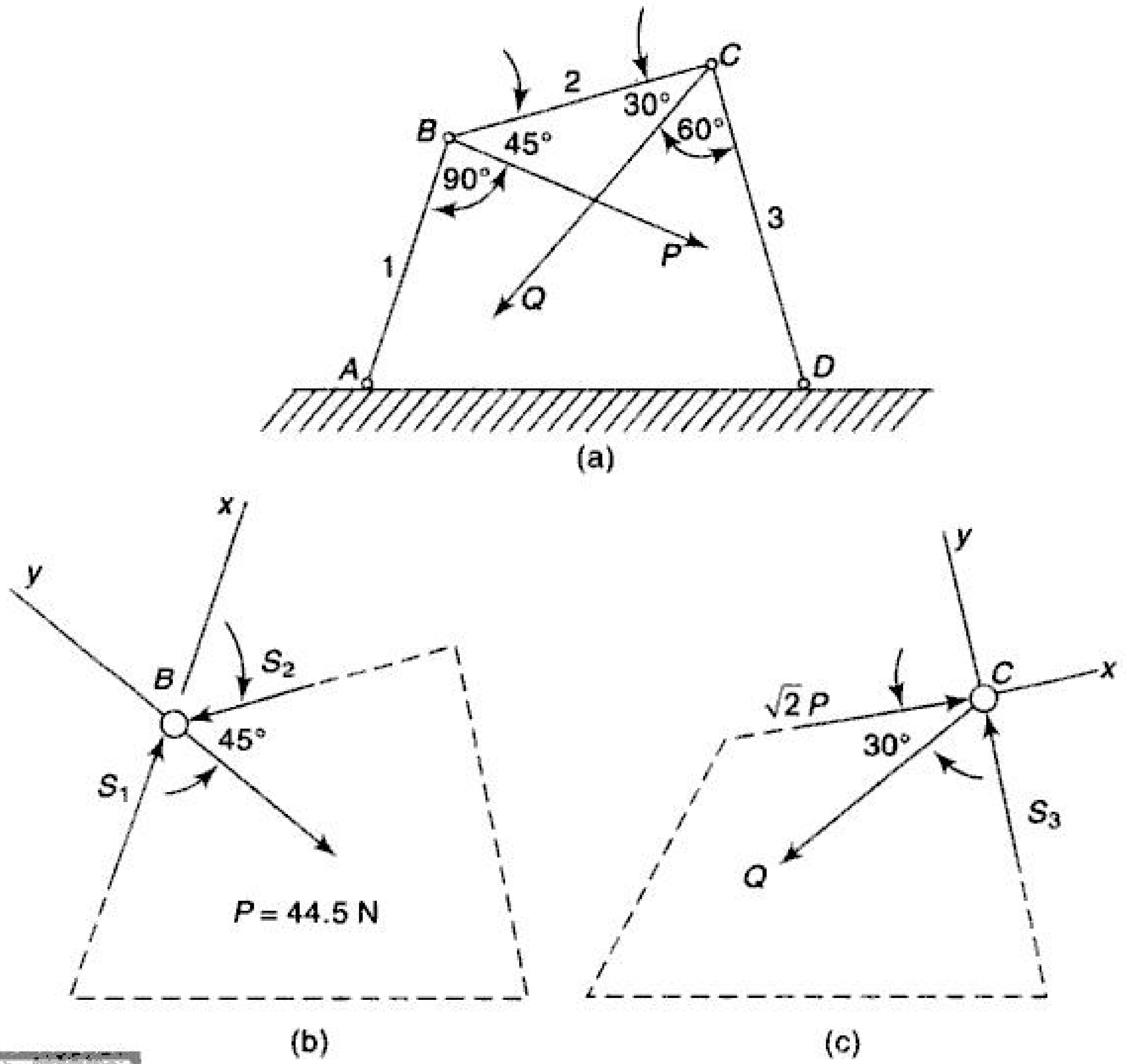


Fig. 2.48

Solution: We begin with a consideration of the equilibrium of the hinge B. Under the action of the applied force P, the bars AB and BC will be subjected to compression and will accordingly exert reactive forces S_1 and S_2 on the pin and with lines of action coinciding with the axes of the bars. All this is shown on the free-body diagram of hinge B in Fig. 2.48(b). Equating to zero the algebraic sum of projections of these forces on the y axis taken perpendicular to AB, we obtain

$$S_2 \cos 45° - P = 0$$

from which $S_2 = P/\cos 45° = \sqrt{2}P$. Since this represents the compressive force in the bar BC, we conclude at once that this bar exerts the same force on the hinge C directed as shown in Fig. 2.48(c).

In addition there is the applied force Q and a reactive force S_3 due to compression in the bar CD. Equating to zero the algebraic sum of the projections of these forces on the x axis perpendicular to CD as shown, we obtain

$$\sqrt{2}P - Q\cos 30^\circ = 0$$

from which $Q = 2\sqrt{2}P/\sqrt{2} = 1.63\ P = 72.54$ N

We see that by choosing one of the coordinate axes in each case perpendicular to the line of action of an unknown force, we obtain an equation of equilibrium containing only one unknown and thereby gain some simplification. If the forces S_1 and S_3 are required, they can easily be found by writing the other two equations of equilibrium.

It is worthwhile to note that if the system of bars in Fig. 2.48(a) is disturbed from the configuration of equilibrium shown, it will collapse. However, if the hinge C is connected to the foundation by another bar AC in place of the force Q, the system will be stable and the magnitude Q calculated above will simply represent the axial force (tension) in such a bar due to the action of the other applied force P.

5. A small ring A can slide without friction along a curved bar CD which has a circular axis of radius a (Fig. 2.49). Determine the position of equilibrium as defined by the angle α if the loads P and Q are acting as shown in the figure.

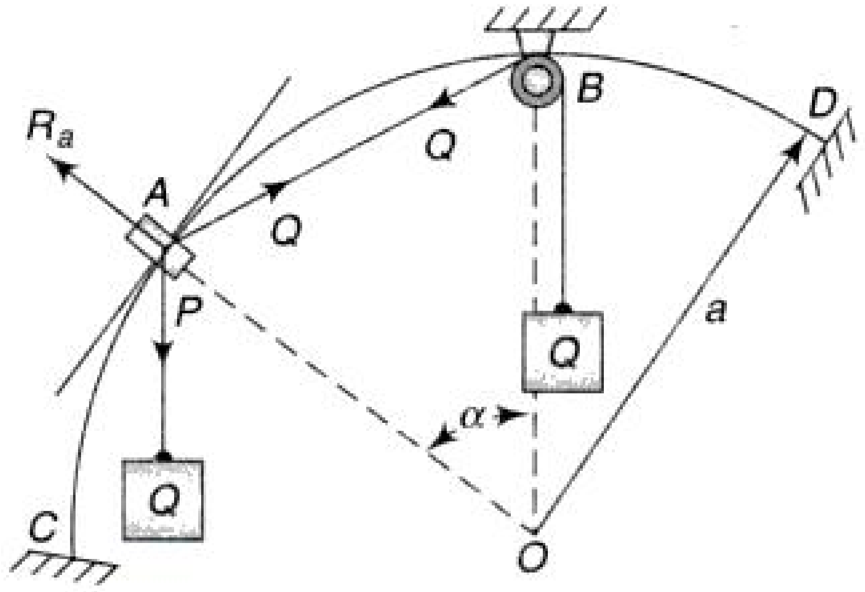


Fig. 2.49

Solution: Neglecting friction in the pulley at B, we conclude that the string AB is subjected to a tension numerically equal to Q. Then considering the ring A as a free body, we see that it is acted upon by the three forces P, Q, and the reaction R_a which acts in the radial direction OA. These three forces are in equilibrium; hence the algebraic sum of their projections on any axis must be equal to zero. Projecting them onto an axis in the direction of the tangent to the circle at A (thus excluding the unknown reaction R_a), we obtain

$$Q\cos\frac{\alpha}{2} - P\sin\alpha = 0$$

which may be written in the form

$$Q\cos\frac{\alpha}{2} - 2P\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 0$$

$$S_4 - Q = 0 \qquad \text{giving } S_4 = Q \text{ tension}$$

As the pulleys at D and E are frictionless, it is evident that the tensions in the string CD and CE are equal to weights P and Q. Thus, acting on the ring C, we have four concurrent forces in a plane that are in equilibrium. The free body diagram of the ring C is shown in Fig. 2.50(d). The forces acting on the ring are string forces 1, 2, 3 and 4. Taking the coordinate axes x and y as shown in the figure, Eq. (3) become

$$S_2 \cos \alpha + Q - S_1 \cos \beta - P \cos \gamma = 0$$

$$S_2 \sin \alpha + S_1 \sin \beta - P \sin \gamma = 0$$

Substituting the values for P and Q; angles α, β and γ in these equations, and solving for S_1 and S_2, we obtain

$$S_1 = 482.46 \text{ N Tension}$$

$$S_2 = 92.82 \text{ N Tension}$$

7. Two smooth cylinders each of weight P and Q, respectively, rest in a horizontal channel having one inclined wall and one vertical wall, the distance between them at bottom which is a [Fig. 2.51(a)]. Find the pressures exerted on the walls and floor at the points of contact A, B and D. The following numerical data are given: $P = 2000$ N and $Q = 800$ N; $r_0 = 100$ mm, $r_2 = 50$ mm and $a = 200$ mm; $\alpha = 60°$.

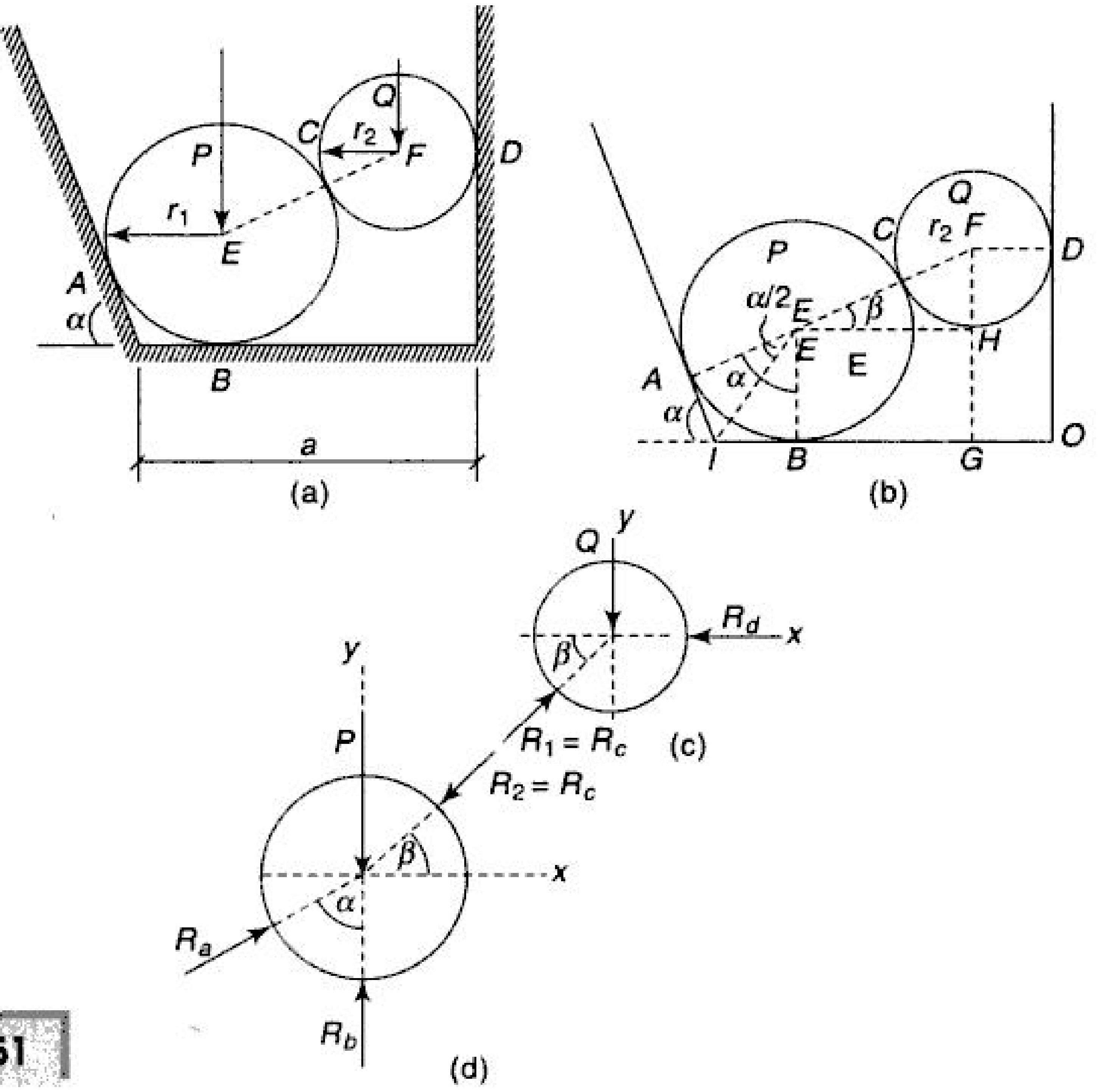


Fig. 2.51

Consider the equilibrium of the lower cylinder now. Then choosing the coordinate axes x and y as shown in Fig. 2.51(d), the equations of equilibrium (3) become

$$R_a \sin\alpha = R_c \cos\beta$$
$$R_a \cos\alpha + R_b = R_c \sin\beta + P$$

Substituting the value of α, β, P and R_c in the above equations, and solving for R_a and R_b we get

$$R_a = R_c \frac{\cos\beta}{\sin\alpha} = 1014.52 \times \frac{\cos 52.05}{\sin 60} = 720.42 \text{ N}$$

$$R_b = R_c \sin\beta + P - R_a \cos\alpha$$
$$= 1014.52 \times \sin 52.05^\circ + 2000 - 720.42 \cos 60^\circ$$
$$R_b = 2439.79 \text{ N}$$

Important Terms and Concepts

Projections of forces Resultant Equations of equilibrium

SUMMARY

- The projections, on the coordinate axes, of the resultant of a system of concurrent forces $F_1, F_2, \ldots, F_n$ acting in one plane are equal to the algebraic sum of the corresponding projections of the components.
- Knowing the magnitudes and directions of the various forces, their projections X_1, $X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ on the rectangular coordinate axes x and y, respectively, may be computed and tabulated in systematic order.

  The algebraic summations, indicated by Eq. (c), for determining the projections X and Y of the resultant may then be made

  $$X = X_1 + X_2 + \ldots\ldots\ldots\ldots + X_n = \Sigma X_i \qquad \text{(c)}$$
  $$Y = Y_1 + Y_2 + \ldots\ldots\ldots\ldots + Y_n = \Sigma Y_i$$

  The summations are understood to include all forces in the system. The magnitude and direction of the resultant computed from the following equations:

  $$R = \sqrt{X^2 + Y^2} \quad \tan\alpha = \frac{Y}{X} \qquad \text{(d)}$$

- When the given forces $F_1, F_2, \ldots, F_n$ are in equilibrium, their resultant is zero, and from the first of Eqs (d), it is evident that this condition can be satisfied only if we have $X = 0$ and $Y = 0$, which, referring to Eqs (c), evidently requires

  $$\Sigma X_i = 0 \qquad \Sigma Y_i = 0 \qquad \text{(e)}$$

Important Formulae

1. The magnitude and direction of the resultant computed from the following equations:

   $$R = \sqrt{X^2 + Y^2} \quad \tan\alpha = \frac{Y}{X}$$

   where X = Algebraic sum of components of forces along x-axis
   Y = Algebraic sum of components of forces along y-axis
   α = the angle made by the resultant with the x-axis
2. The two equations of equilibrium for a system of concurrent forces in a plane are

   $$\Sigma X_i = 0 \qquad \Sigma Y_i = 0$$

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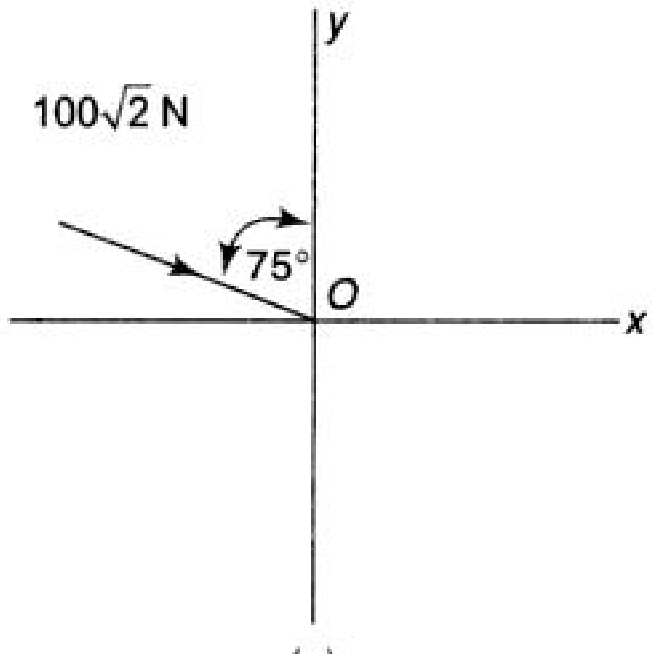
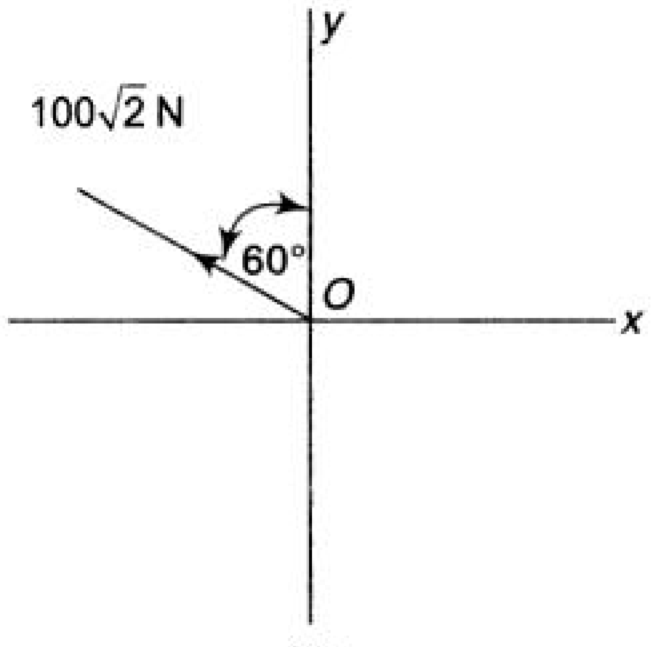


(c)



(d)

[*Ans.* (c)]

4. A lamp of weight $W = 100$ N is supported by two cables CA and CB as shown in the Fig. C. The equation for analysing the cable system is given by

(a) $T_{CA}\dfrac{2}{\sqrt{4.563}} + T_{CB}\dfrac{1.5}{\sqrt{2.813}} = 0$ (b) $T_{CA}\sin\alpha + T_{CA}\sin\beta + 100 = 0$

(c) $T_{CB}(3.5)\sin\alpha = 100(1.5)$ (d) $T_{CA}(3.5)\sin\alpha = 100(1.5)$ [*Ans.* (a)]

PROBLEM SET 2.4

1. Using the method of projections, find the magnitude and direction of the resultant R of the four concurrent forces shown in Fig. A and having the magnitudes $F_1 = 1500$ N, $F_2 = 2000$ N, $F_3 = 3500$ N and $F_4 = 1000$ N.

(*Ans.* $R = 1842.6$ N and $\alpha = 227°$)

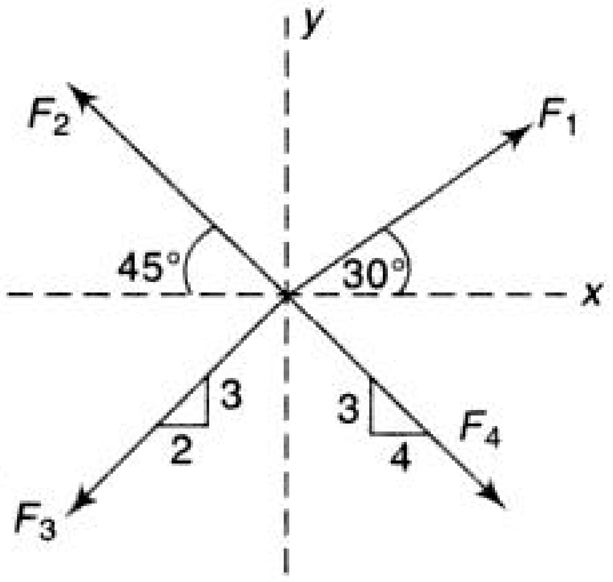


Fig. A

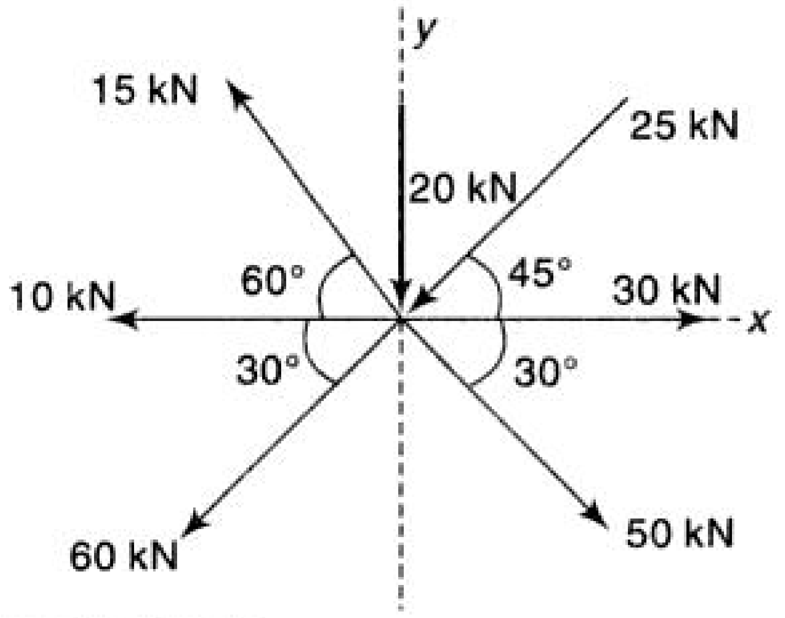


Fig. B

2. Forces of 2, 3, 4, 5 and 6 kN are acting at one of the angular points of a regular hexagon towards the other angular points taken in order. Find the resultant of the system of forces. (*Ans.* $R = 15.6$ kN; $\alpha = 76.7°$)

3. Find the magnitude and direction of the force F to be added to the system of coplanar concurrent forces shown in Fig. B to maintain equilibrium.

(*Ans.* $F = 91$ kN, $\alpha = 61.18°$)

4. Referring to Fig. C, calculate the tensions S_1 and S_2 in the two strings AB and AC that support the lamp of weight $Q = 178$ N. Use the method of projections [Eq (3)].

(*Ans.* $S_1 = 133.5$ N; and $S_2 = 222.5$ N)

12. Determine the axial forces S_1 and S_2 induced in the bars *AC* and *BC* in Fig. K due to the action of the horizontal applied load at *C*. The bars are hinged together at *C* and to the foundation at *A* and *B*

(*Ans.* $S_1 = 3475$ N, tension; $S_2 = 2849$ N, compression)

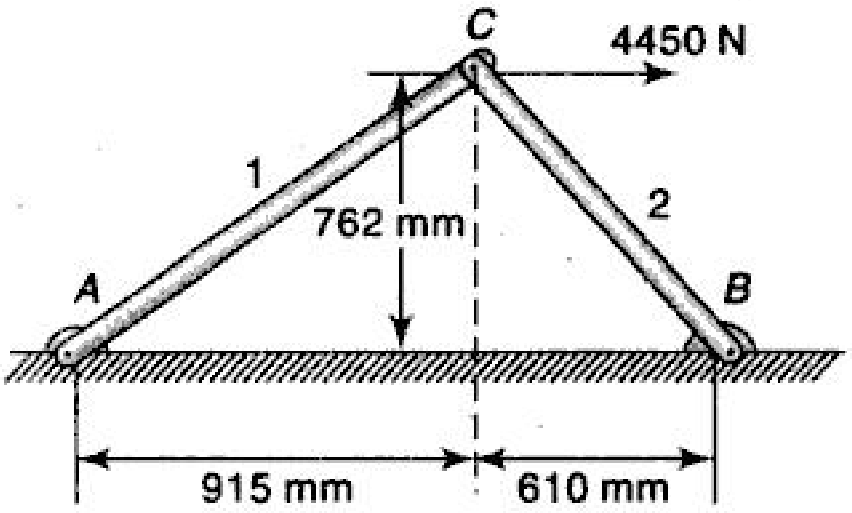


Fig. K

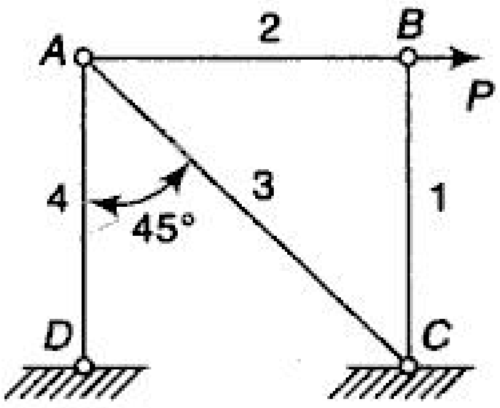


Fig. L

13. Determine the forces produced in the bars of the system shown in Fig. L owing to the horizontal force *P* applied at the hinge *B*.

(*Ans.* $S_1 = 0$; $S_2 = P$, tension; $S_3 = \sqrt{2P}$, compression; $S_4 = P$, tension)

14. A hinged square *ABCD* (Fig. M) with diagonal *BD* is submitted to the action of two equal and opposite forces applied as shown. Determine the forces produced in all bars. (*Ans.* $S_1 = S_2 = S_3 = S_4 = S_5 = 0$; $S_3 = P$, tension)

15. Determine the forces that will be produced in all bars of the frame *ABCD* (Fig. M) if the external forces are applied in the same manner to the hinges *A* and *C*.

(*Ans.* $S_1 = S_2 = S_3 = S_4 = S_5 = P/\sqrt{2}$, tension; $S_3 = P$, compression)

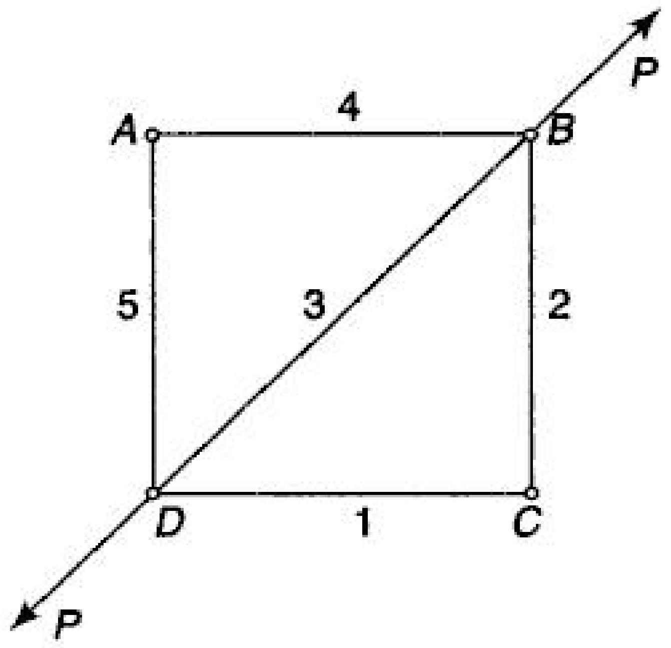


Fig. M

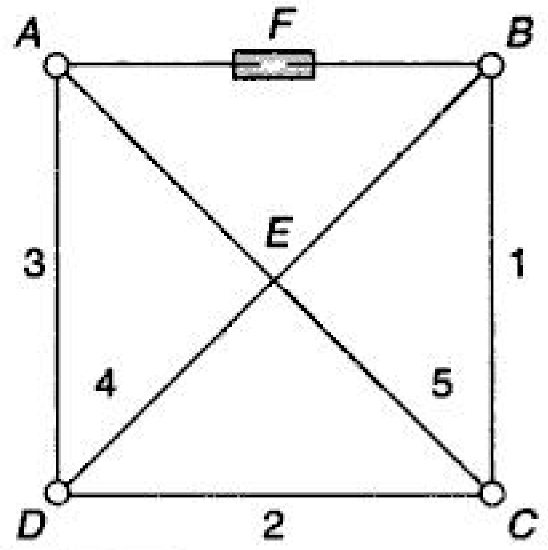


Fig. N

16. In the bar of the square frame *ABCD* (Fig. N) a tensile force *P* is produced by tightening a turnbuckle *F*. Determine the force produced in the other bars. The diagonals *AC* and *BD* pass each other freely at *E*.

(*Ans.* $S_1 = S_2 = S_3$ *P*, tension; $S_4 = S_5 = \sqrt{2}P$, compression)

17. By means of a turnbuckle *A* a tensile force *P* is produced in one of the radial bars of the hinged regular octagon shown in Fig. O. Determine the forces produced in the other bars of the system.

(*Ans.* *P*, tension in each radial bar; 1.306 *P*, compression in each outside bar)

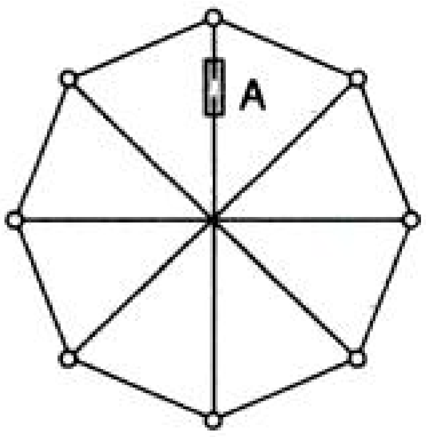


Fig. O

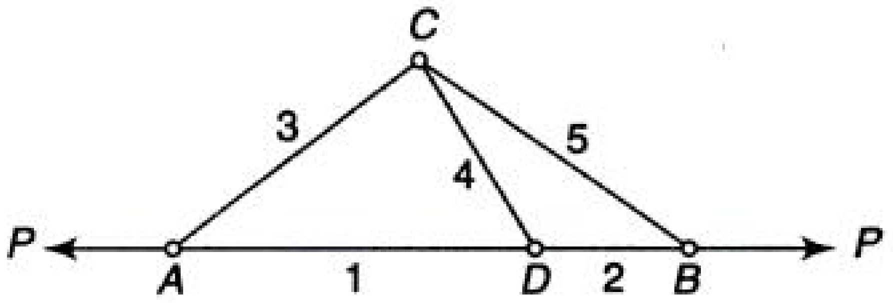


Fig. P

18. Determine the axial force induced in each bar of the system shown in Fig. P due to the action of the applied forces P. (*Ans.* $S_1 = S_2 = P$ tension; $S_3 = S_4 = S_5 = 0$)
19. The smooth cylinders rest in a horizontal channel having vertical walls, the distance between which is a (Fig. Q). Find the pressures exerted on the walls and floor at the points of contact A, B, D and F. The following numerical data are given: $P = 200$ N, $Q = 400$ N, $R = 300$ N, $r_1 = 120$ mm, $r_2 = 180$ mm, $r_3 = 150$ mm and $a = 540$ mm. (*Ans.* $R_a = 525$ N, $R_b = 900$N, $R_d = 772.48$ N and $R_f = 247.48$ N)

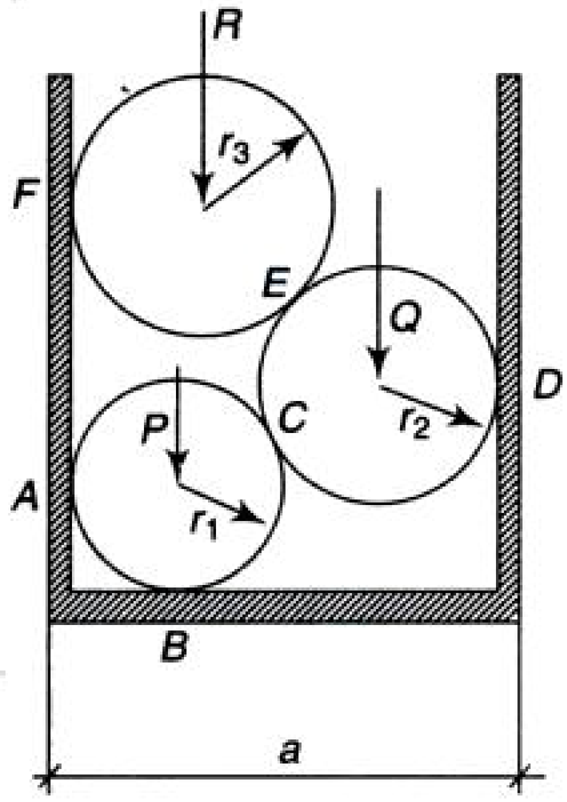


Fig. Q

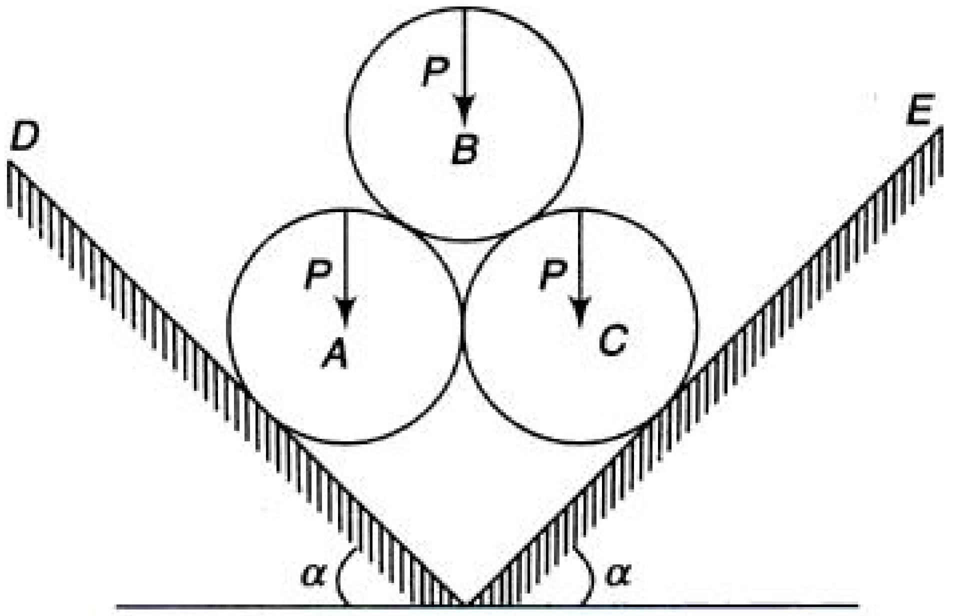


Fig. R

20. In Fig. R, three smooth right circular cylinders, each of radius r and weight P, are arranged on smooth inclined surfaces as shown. Determine the least value of angle α that will present the arrangement from slipping. (*Ans.* $\alpha = 10.\ 9°$)
21. Two smooth cylinders of weights P and Q are placed in a smooth trough as shown in Fig. S. Determine the reactions at contact surfaces A, B and C. The following numerical data are given: $P = 200$ N and $Q = 800$ N; $r_1 = 100$ mm, $r_2 = 200$ mm, and $a = 400$ mm; $\alpha = 45°$. (*Ans.* $R_a = 70.7$ N; $R_b = 1414.21$ N, $R_c = 1070.71$ N)
22. Three smooth spheres of weights P, P and Q are placed in a smooth trench as shown in Fig. T. Find the pressures exerted on the walls and floor at the points of contact A, B, C and D. The following numerical data are given: $P = 0.3$ kN, $Q = 0.6$ kN and $R = 0.3$ kN; $r_1 = 0.4$ m, $r_2 = 0.6$ m and $r_3 = 0.4$ m; $\alpha = 30°$. (*Ans.* $R_a = 61$N, $R_b = 635$ N, $R_c = 1100$N, $R_d = 291$ N)

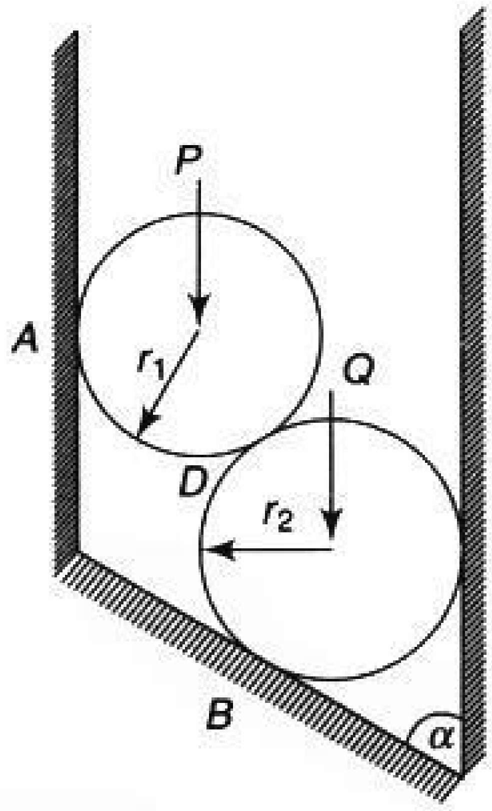


Fig. S

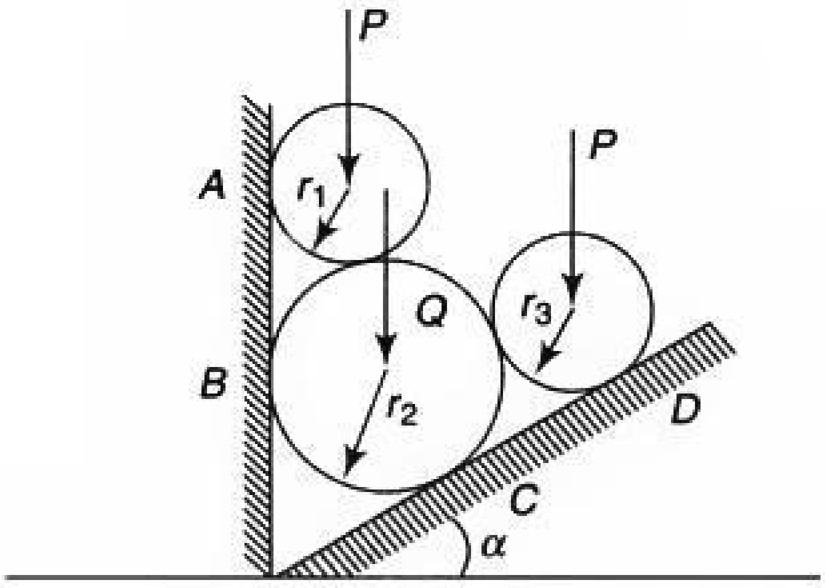


Fig. T

*23. A rigid bar with rollers of weights $P = 222.5$ N and $Q = 445$ N at its ends is supported inside a circular ring in a vertical plane as shown in Fig. U. The radius of the ring and the length AB are such that the radii AC and BC form a right angle at C; that is, $\alpha + \beta = 90°$. Neglecting friction and the weight of the bar, find the configuration of equilibrium as defined by the angle $(\alpha - \beta)/2$ that makes with the horizontal. Find also the reactions R_a and R_b and the compressive force S in the bar.

(*Ans.* $(\alpha - \beta)/2 = 18° \, 26'$; $R_a = 298.5$ N; $R_b = 597$ N; $S = 281.5$ N)

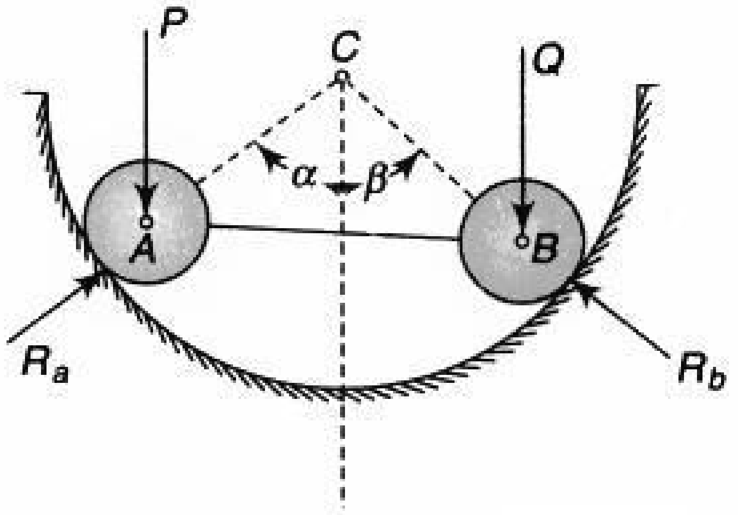


Fig. U

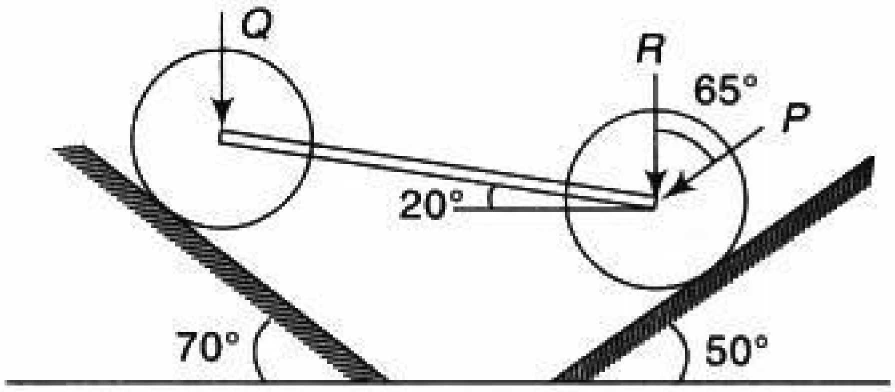


Fig. V

24. Two cylinders of weights Q and R are interconnected by a bar of negligible weight hinged to each cylinder at its geometric center by ideal pins. Determine the magnitude of P applied at the center of cylinder R to keep the cylinders in equilibrium in the position shown in Fig. V. The following numerical data are given: $Q = 2000$ N and $R = 1000$ N (*Ans.* $P \cong 242$ N)

2.5 EQUILIBRIUM OF THREE FORCES IN A PLANE

If three non-parallel forces acting in one plane are in equilibrium, their lines of action must intersect at one point. To prove this statement, let us assume that the three forces P, Q and S, acting upon a body at points A, B and C [Fig. 2.52(a)] all

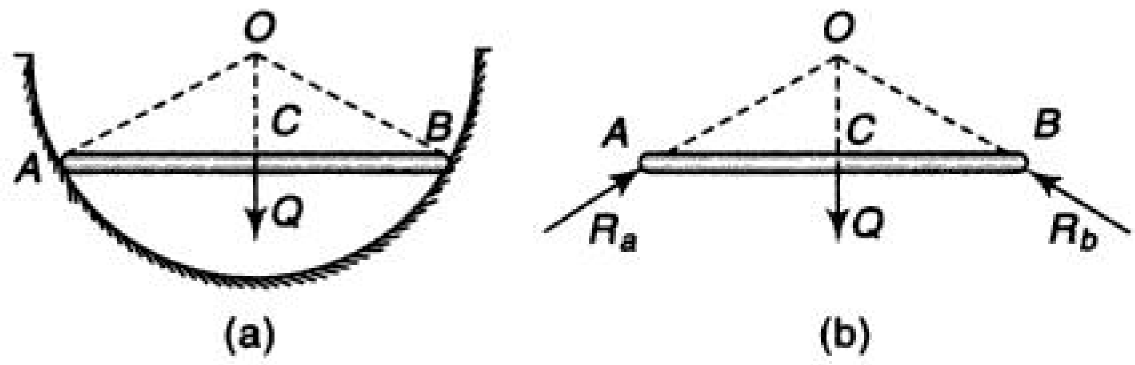


Fig. 2.53

As a second example, consider the case of a lever *AB* supported by a *hinge C*, as shown in Fig. 2.54(a). Under the action of applied forces *P* and *Q* (assumed such as to maintain equilibrium), the lever exerts pressure on the hinge pin which passes through it. We are concerned with the equal and opposite reactive pressure exerted by the pin on the bar. A detail of the pin and hole is shown in Fig. 2.54(b), where we again make the assumption of a perfectly smooth circular cylindrical surface for the pin. Under such conditions, the distributed pressure at all points of contact between pin and inner surface of hole is normal to the surface, as shown. This means that all these distributed pressures must act along radial lines intersecting at the center of the pin. Consequently, the reaction R_c must pass through the center of the pin if it is to produce sensibly the effect of the distributed pressure. This notion of a smooth circular pin or *ideal hinge* will often be used and always the reaction that it produces on a body will pass through its center. Regarding the direction of R_c there is nothing about the physical nature of the constraint to determine this; it can have any line of action through point *C*. However, for equilibrium, we conclude from the theorem of three forces that R_c must pass through the point of intersection *D* of the active forces *P* and *Q* and the free-body diagram is obtained, as shown in Fig. 2.54(c).

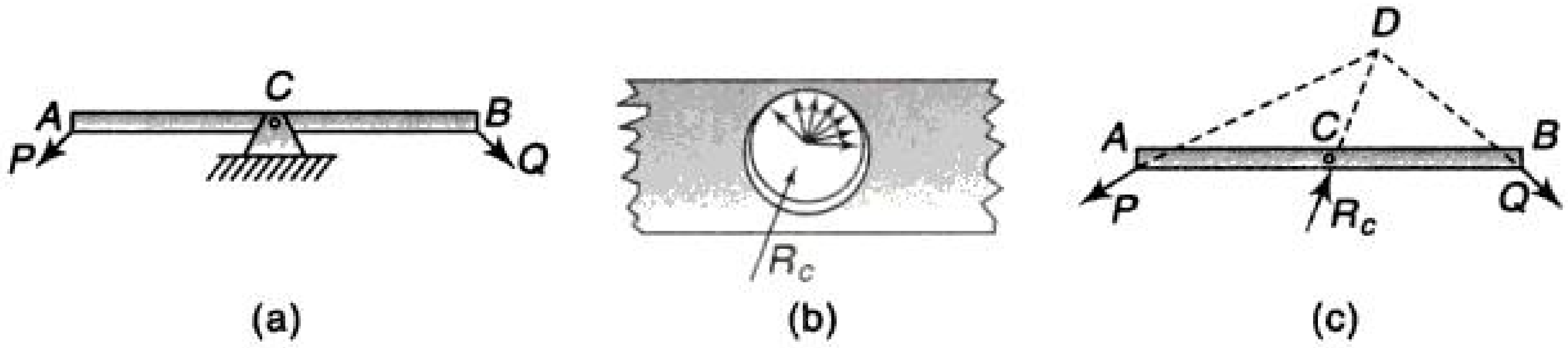


Fig. 2.54

As a last example, we consider the case of a beam *AB* supported horizontally by a hinge at *A* and a small roller at *B*, as shown in Fig. 2.55(a). Under the action of an applied force *P* acting in the plane of the figure as shown, the beam exerts pressures on the two supports at its ends. We wish to find the corresponding reactions. To do this, we remove both supports and replace them by reactions, as shown in Fig. 2.55(b). Assuming an ideal hinge at *A*, the reaction R_a must act through this point, but we do not know immediately in what direction. Consequently we indicate this force by a wavy line as shown, to indicate unknown direction. Now proceeding to point *B* and assuming a smooth surface for the roller, we conclude that R_b must act in a direction normal to the surface on which

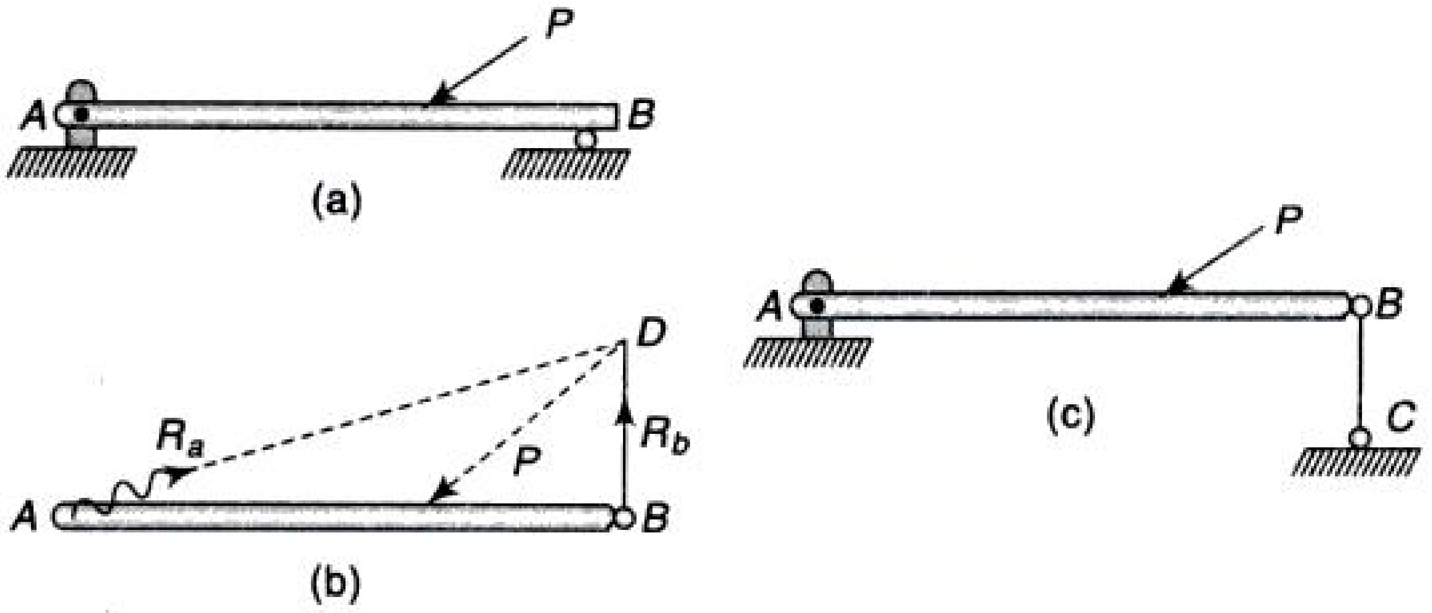


Fig. 2.55

the roller rests; in this case, vertically. Such rollers allowing free horizontal movement of one end of a beam are often used for bridge supports to prevent damage due to contraction or expansion of the beam resulting from changes in temperature. This *simple roller* is a common type of constraint and always exerts its reaction normal to the surface on which it rolls. To complete the free-body diagram in Fig. 2.55(b), we now observe that the known lines of action of P and R_b determine the point of concurrence D of the system of three forces in equilibrium; hence the true line of action AD of the reaction R_a is finally determined.

If the roller at B in Fig. 2.55(a) is replaced by a vertical bar BC of negligible weight and hinged at both ends [Fig. 2.55(c)], we shall obtain the same free-body diagram as before. Since forces act on this bar only at its two ends, their lines of action must coincide with the axis BC, as previously explained. Consequently, the reactive forces that the bar exerts on the beam must be directed along BC. Such a *simple strut* (or *tie bar* if it is in tension) is another common type of constraint used as a support.

Examples Examples Examples Examples Examples

1. The vertical axis AB of a crane is supported by a guide at A and a socket at B as shown in Fig. 2.56(a). Determine the reactions R_a and R_b produced at A and B by the load $P = 35.6$ kN. Friction at the supports should be neglected.

   Solution: Considering the entire crane as a free body [Fig. 2.56(a)], we imagine the supports at A and B removed and replace them by the reactions R_a and R_b which they exert on the crane. Thus we have the case of equilibrium of three forces, P, R_a and R_b in a plane and they must intersect in one point. Strictly speaking, the reaction exerted by the guide at A will be distributed over the surface of contact between the guide and the mast, but since the dimensions of this surface are small compared with the dimensions of the entire structure, we can assume this pressure to be concentrated at one point, say, the center of the guide, and further, neglecting friction, it

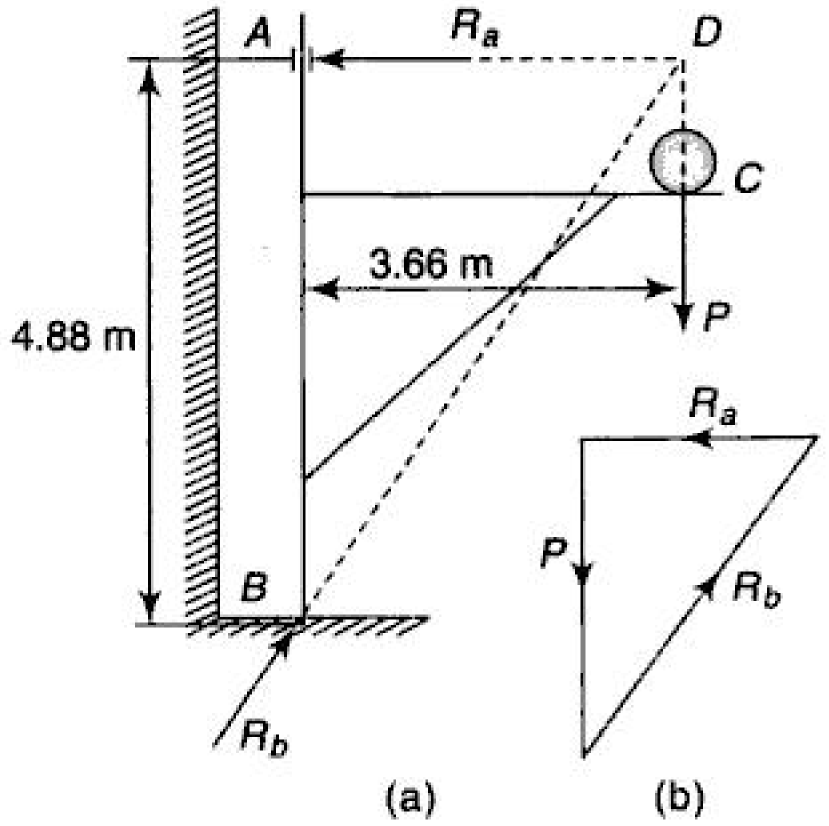


Fig. 2.56

must act perpendicular to AB. Thus by justifiable simplifying assumptions, we obtain the point of application and the direction of the reaction R_a. Extending this line of action to intersect that of the vertical gravity force P, we obtain the point of intersection D, through which the reaction R_b must also pass and its line of action BD is determined.

We now construct the triangle of forces [Fig. 2.56(b)] with its sides parallel to the known lines of action of the three forces. Noting that the triangle of forces is similar, by construction, to ΔABD we conclude that R_a: 3.66 = P : 4.88, from which $R_a = \frac{3}{4}P = 26.7$ kN. In the same way we note that R_b: 6.1 = P: 4.88, from which $R_b = \frac{5}{4}P = 44.5$ kN. We see further, from the directions of the arrows on the sides of the triangle of forces, that the forces act as shown in Fig. 2.56(a).

2. Determine the magnitude of a horizontal force P applied at the center C of a roller of weight $Q = 4450$ N and radius $r = 381$ mm which will be necessary to pull it over a 76.2 mm-curb. [Fig. 2.57(a)].

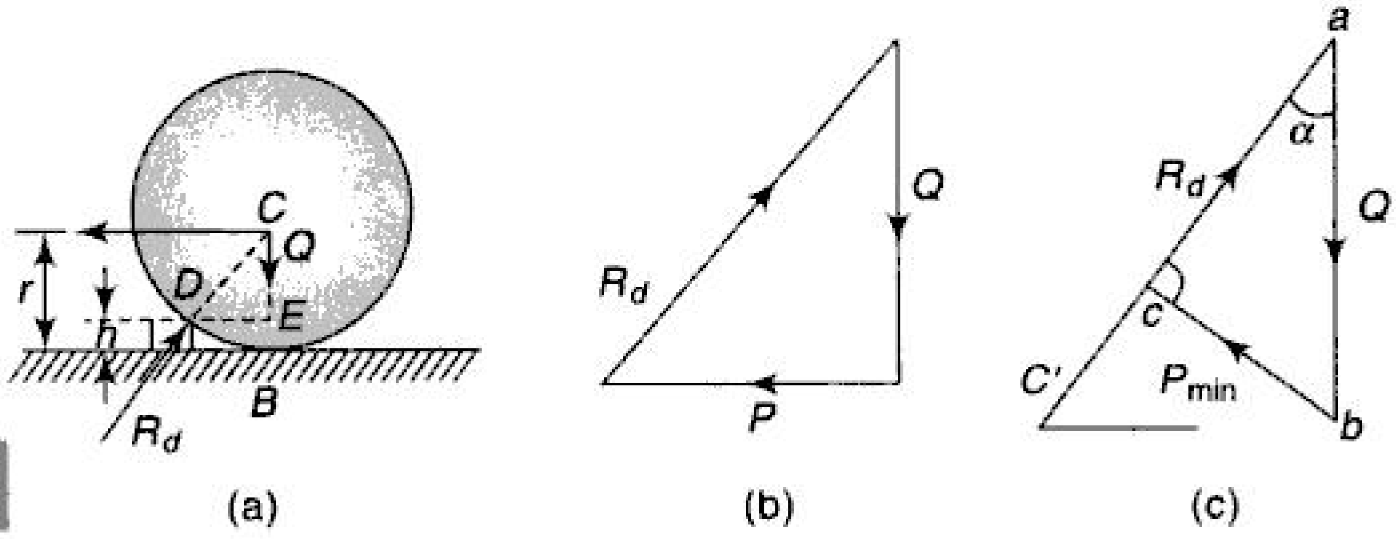


Fig. 2.57

- The simple roller is a common type of constraint and always exerts its reaction normal to the surface on which it rolls.
- The simple bar is another common type of constraint used as a support. The reactive forces that the bar exerts on the body must be directed along the bar. A bar under tension is called tie. A bar under compression is called a strut.

PRACTICE SET 2.5

Review Questions

1. State the theorem of three forces in a plane.
2. Explain the reactions exerted by the supports namely roller support, hinge support, simple bar.
3. The reaction from an ideal smooth surface must be directed along the ______ at the point of contact.
4. Sketch the different types of supports and the reactions developed in each type.

Objective Questions

1. Consider the following statements.
   I. Three nonparallel forces can be in equilibrium only when they lie in one plane, intersect in one point, and their free vectors build a closed triangle.
   II. The reaction from an ideal smooth surface must be directed along the normal at the point of contact.
   Of these statements
   (a) I alone is correct (b) II alone is correct
   (c) I and II are correct (d) Neither I nor II is correct [*Ans.* (c)]

PROBLEM SET 2.5

1. A boat is suspended on two identical davits like *ABC* which is pivoted at *A* and supported by a guide at *B* (Fig. A). Determine the reactions R_a and R_b at the points of support *A* and *B* if the vertical load transmitted to each davit at *C* is 4272 N. Friction in the guide at *B* should be neglected.
   (*Ans.* $R_a = 7121.73$N; $R_b = 5696.87$ N)
2. A prismatic bar *AB* of weight $Q = 17.8$ kN is hinged to a vertical wall at *A* and supported at *B* by a cable *BC* (Fig. B). Determine the magnitude and direction of the

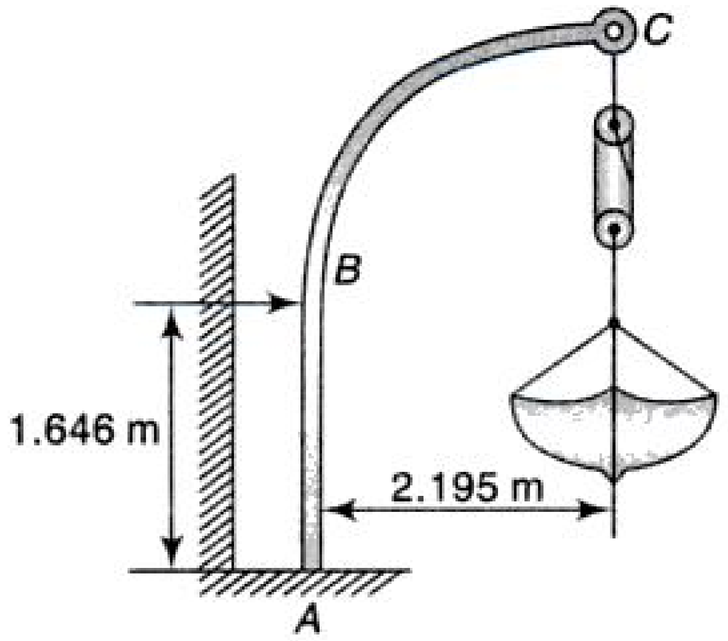


Fig. A

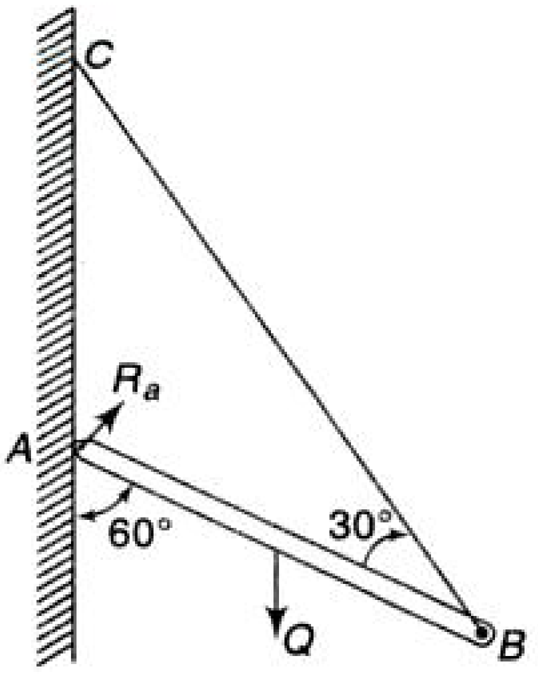


Fig. B

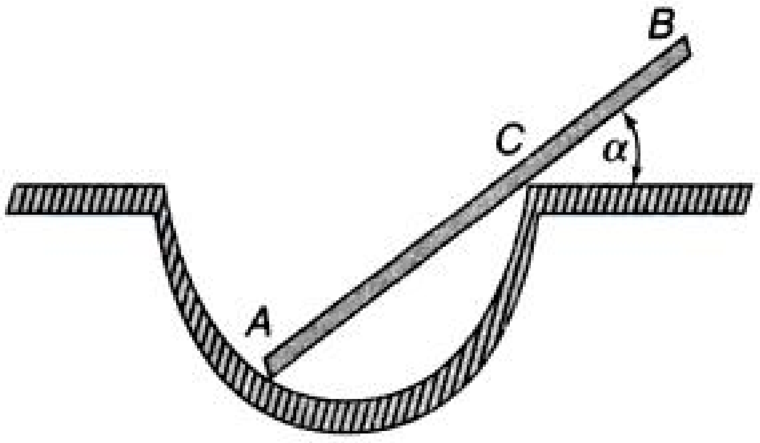


Fig. O

2.6 METHOD OF MOMENTS

Moment of a Force with Respect to a Point. The concept that a force tends to produce rotation about a fixed point in any body to which it is applied is useful in the solution of problems of statics. Consider, for example, the wrench shown in Fig. 2.59, to the handle of which two equal forces P and Q are applied as shown. It is a matter of common experience that the force P acting at right angles to the handle is more effective in tending to turn the nut to which the wrench is fitted than is the force Q, even though they are of equal magnitude. The effectiveness or importance of a force, as regards its tendency to produce rotation of a body about a fixed point, is called the *moment* of the force with respect to that point, and this moment can be measured by the product of the magnitude of the force and the distance from the point to the line of action of the force. Thus, the magnitude of the moment of the force Q with respect to the point O (Fig. 2.59) depends not only upon the magnitude of the force itself but also upon the distance OD from the point O to its line of action. The point O is called the *moment center*, and the distance OD is called the *arm of the force.*

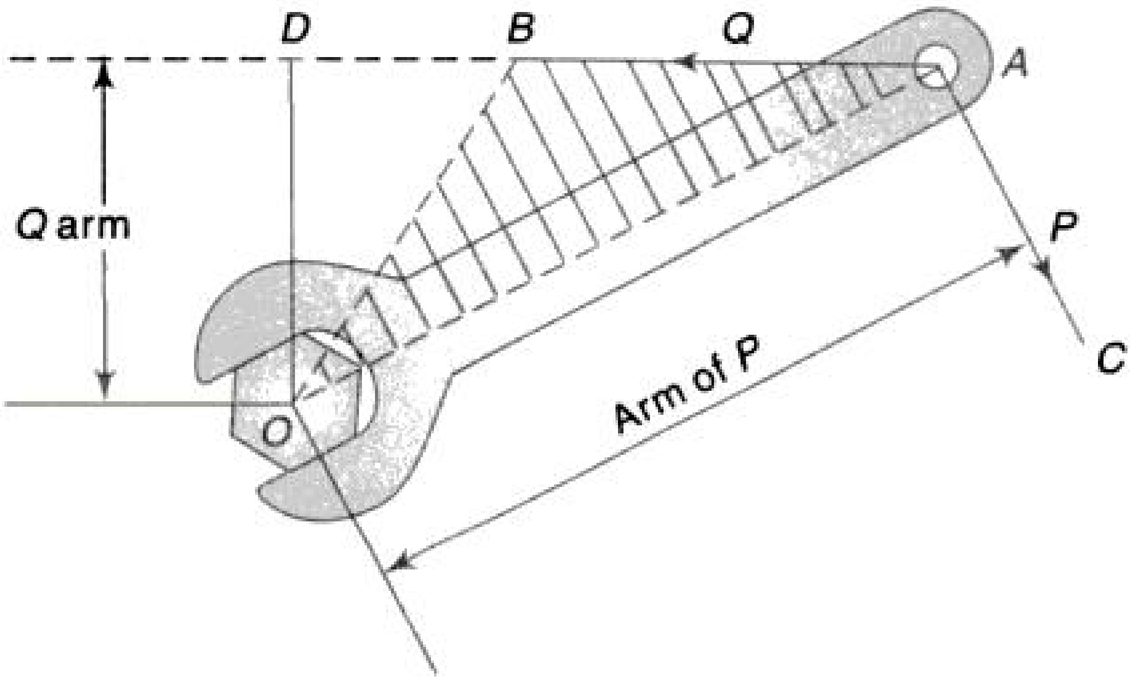


Fig. 2.59

From the preceding definition, it follows that the moment of the force Q is numerically equal to the doubled area of ΔABO, constructed on the vector $\overline{AB}$ representing the force, and having its vertex at the moment center. In this calculation the vector $\overline{AB}$ should be measured to the scale used for representing force, while the arm OD should be measured to the scale used for length. Thus it is seen that the unit of moment of force is the unit of length times the unit of force. For example, taking the *newton* as the unit of force and the *metre* as the unit of length,

from which

$$R_b = \frac{l}{2} Q \cos\alpha$$

2. Figure 2.64 represents the cross-section of a retaining wall supporting an earth fill. The earth pressure per meter of length of wall can be replaced by its resultant H = 58.4 kN/m acting as shown in the figure. Determine the minimum thickness b of the wall required to prevent overturning about the front edge A if h = 4.57 m and the specific gravity of the wall is $2\frac{1}{2}$.

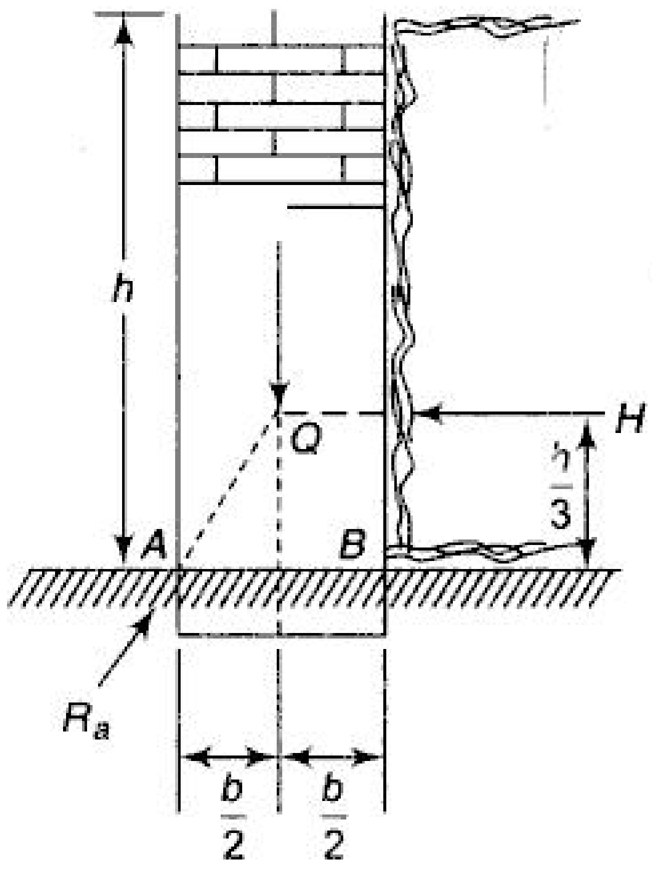


Fig. 2.64

Solution: Let us assume that we are dealing with a unit section of the wall 1 m in length normal to the plane of the figure. If such a section is made stable, then a similar wall of any length will be equally stable. When conditions are such that overturning of the wall impends the reaction R_a exerted by the foundation on the bottom of the wall will be concentrated at the front edge A as shown in the figure. Thus the 1-m section of wall is in equilibrium under the action of three forces: the weight

$$Q = 2\frac{1}{2} \times 9.8 \times 4.57b = 112b \text{ kN}$$

acting at the center of the cross-section, the earth pressure

$$H = 58.4 \text{ kN}$$

and the reaction R_a. The algebraic sum of the moments of these forces with respect to any center in the plane of the cross-section must be equal to zero. Taking point A as the moment center, we obtain

$$H\frac{h}{3} - Q\frac{b}{2} = 0$$

Substituting the numerical data as given and solving for b gives

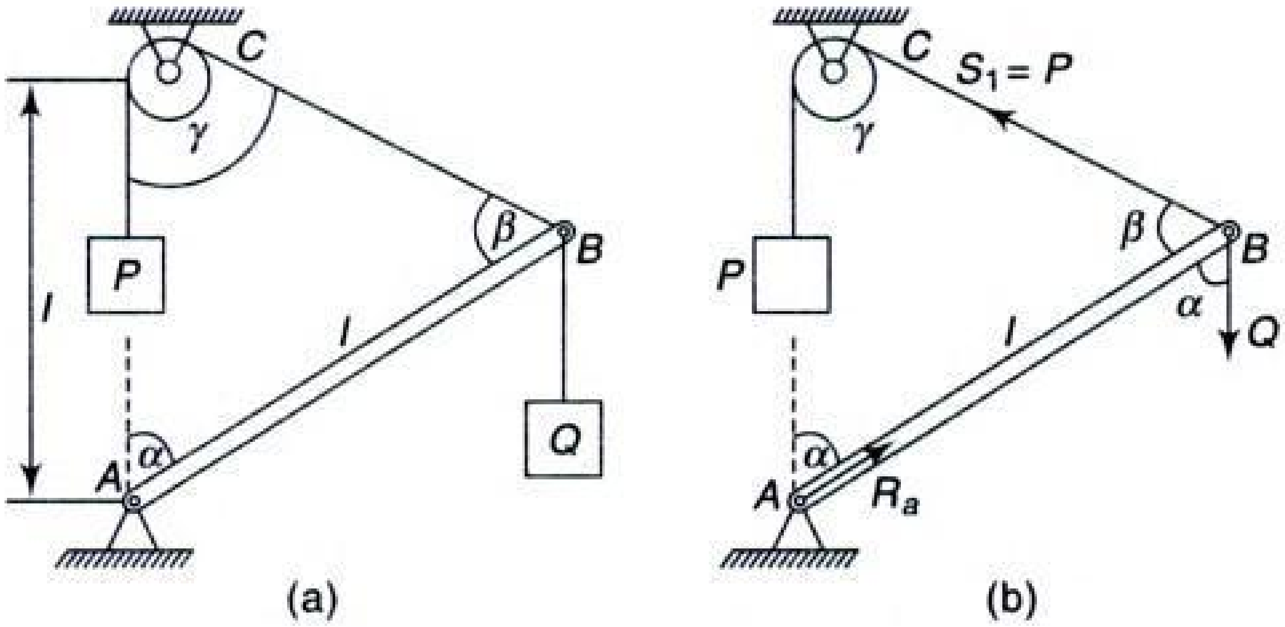
$$b = \sqrt{1.589} = 1.261 \text{ m}$$

3. A slender prismatic bar AB of weight Q and length $2l$ rests on a very small frictionless roller at D and against a smooth vertical wall at A, as shown in Fig. 2.65. Find the angle α that the bar must make with the horizontal in the condition of equilibrium.

Solution: Isolating the bar AB, we obtain the free-body diagram as shown in the figure. The reaction at A is normal to the wall, that is, horizontal, and the reaction at D is normal to AB. When the bar is in a condition of equilibrium, the three forces Q, R_a and R_d meet in one point and the algebraic sum of their projections on any axis must be zero [Eq. (3)]. Likewise, the alge-

6. A prismatic bar *AB* of negligible weight and length *l* is hinged at *A* and supported at *B* by a string that passes over a pulley *C* and carries a load *P* at its free end (Fig. 2.68). Assuming that the distance *h* between the hinge *A* and the pulley *C* is equal to the length *l* of the bar, find the angle α at which the system will be in equilibrium.

Fig. 2.68



Solution: We begin with a consideration of the equilibrium of the bar *AB* which is acted upon by three forces: A force *Q* representing the pull of the string at *B*, a force *P* representing the pull of the string at *B*, and a reaction at the hinge *A*. As the forces *P* and *Q* meet at *B*, we conclude that the reaction *Ra* must be directed along the line *AB* as shown.

The triangle *ABC* is isosceles

$$\beta = \gamma = \frac{\pi - \alpha}{2} = 90^\circ - \left(\frac{\alpha}{2}\right)$$

Taking point *A* as the moment center (thus eliminating consideration of the unknown reaction at *A*), we obtain

$$(l)P \sin\left(90^\circ - \frac{\alpha}{2}\right) - (l)(Q \sin \alpha) = 0$$

$$P \cos\frac{\alpha}{2} - (Q \sin \alpha) = 0$$

$$P \cos\frac{\alpha}{2} - \left(Q \times 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}\right) = 0$$

$$\Rightarrow \qquad \sin\frac{\alpha}{2} = \frac{P}{2Q}$$

$$\Rightarrow \qquad \alpha = 2 \sin^{-1}\left(\frac{P}{2Q}\right)$$

Important Terms and Concepts

Moment of a force with respect to a point
Moment center
Arm of the force
Newton-metre
Theorem of Varignon
Equilibrium equations

7. A vertical load P is supported by a triangular bracket as shown in Fig. G. Find the forces transmitted to the bolts A and B. Assume that the bolt B fits loosely in a vertical slot in the plate. (*Ans.* $R_a = 1.25P$; $R_b = 0.75\,P$)

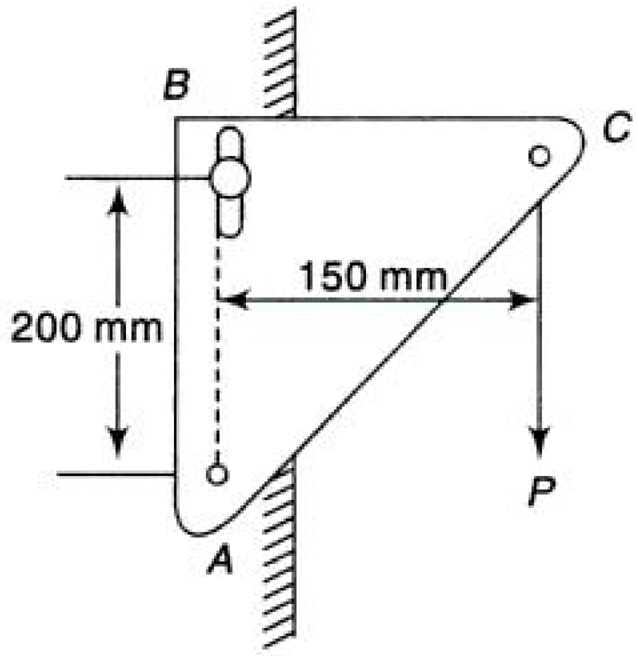


Fig. G

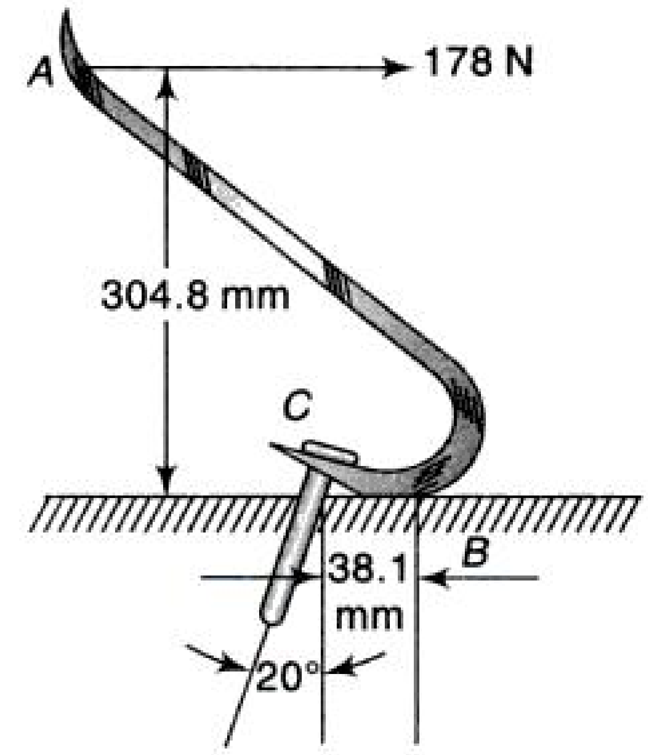


Fig. H

8. Find the magnitude of the pull P exerted on the nail C in Fig. H if a horizontal force of 178 N is applied to the handle of the wrecking bar as shown. (*Ans.* $P = 1515.4$ N)
9. Determine the forces exerted on the cylinder at B and C by the spanner wrench shown in Fig. I due to a vertical force of 222.5 N applied to the handle as shown. Neglect friction at B. (*Ans.* $R_b = 1068$ N; $R_c = 1091$ N)

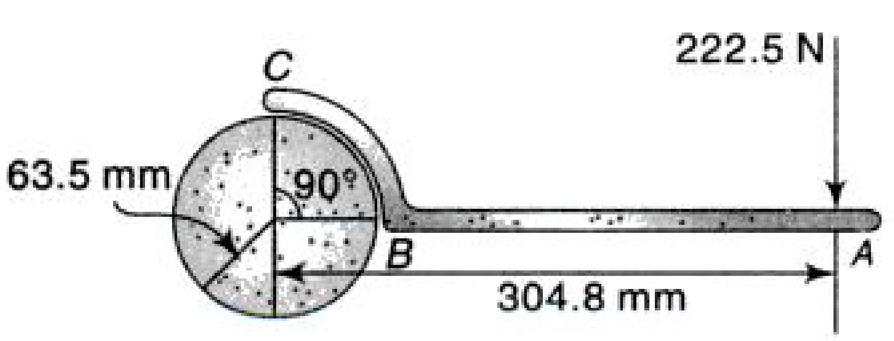


Fig. I

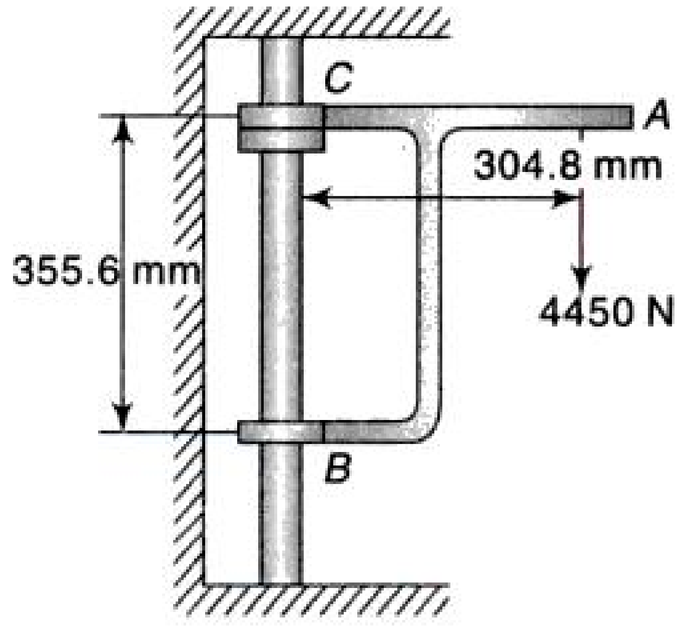


Fig. J

10. A bracket ACB can slide freely on the vertical shaft BC but is held by a small collar attached to the shaft as shown in Fig. J. Neglecting all friction, find the reactions at B and C for the vertical load shown. (*Ans.* $R_b = 3814.3$ N; $R_c = 5861$ N)
11. Two beams AB and DE are arranged and supported as shown in Fig. K. Find the magnitude of the reaction R_e at E due to the force $P = 890$ N applied at B as shown. (*Ans.* $R_e = 445$ N)
12. A smooth right circular cylinder of radius r rests on a horizontal plane and is kept from rolling by an inclined string AC of length $2r$ (Fig. L). A prismatic bar AB of length $3r$ and weight Q is hinged at point A and leans against the roller as shown. Find the tension S that will be induced in the string AC. (*Ans.* $S = 0.433\,Q$)

These laws of friction may be expressed by the simple formula,

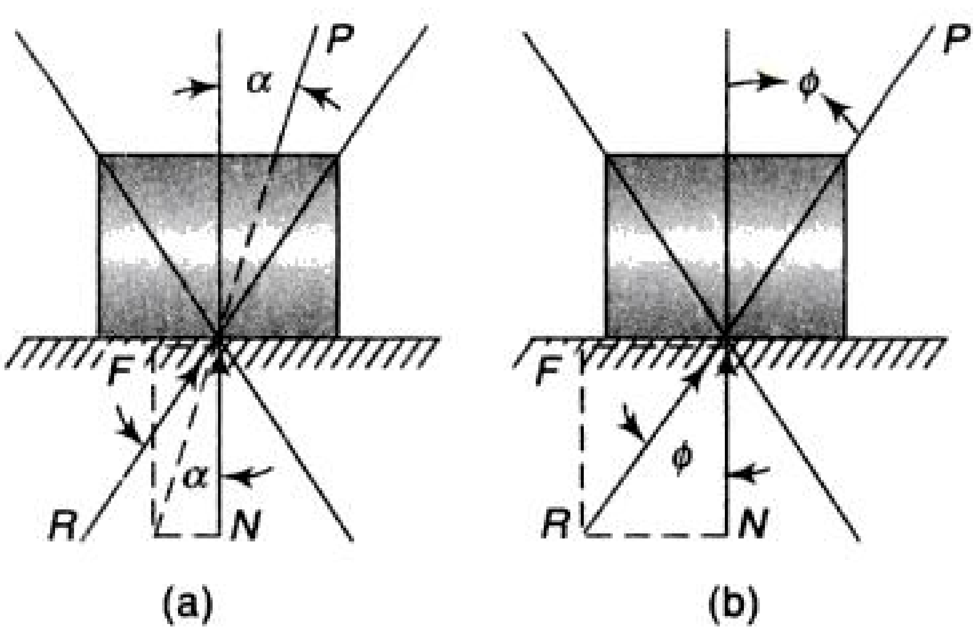
$$F = \mu N \tag{7}$$

where μ is called the coefficient of friction. If F is taken as the force necessary to start sliding, μ is called the coefficient of static friction. If F is taken as the somewhat smaller force necessary to maintain sliding, once it has been started, μ is called the coefficient of kinetic friction. The coefficients of static and kinetic friction vary greatly for different materials and for different conditions of their surfaces. Table 2.1 lists approximate values of coefficients of friction for various materials.

Table 2.1 *Coefficients of Friction*

Materials	*Static friction*		*Kinetic friction*	
	μ	φ^0	μ	φ^0
Leather on wood	0.5 – 0.6	27 – 31	0.3 – 0.5	17 – 27
Leather on metal	0.3 – 0.5	17 – 27	About 0.3	About 17
Masonry on dry clay	About 0.5	About 27		
Metal on metal	0.15 – 0.25	8 – 14	About 0.1	About 6
Metal on wood	0.4 – 0.6	22 – 31	0.3 – 0.5	17 –27
Rope on wood	0.5 – 0.8	27 – 39	About 0.5	About 27
Stone on stone	0.6 – 0.7	31 – 35		
Stone on wood	About 0.4	About 22		
Wood on wood	0.4 – 0.7	22 – 35	About 0.3	About 17
Steel on ice	About 0.03	About 2	0.015	About 1

To see how friction affects the reactions exerted by supporting surfaces, let us consider the simple case of a small block resting upon a horizontal plane surface and acted upon by a force P making the angle α with the vertical [Fig. 2.70(a)]. We shall assume, for simplicity, that the force P is large in comparison with the weight of the block so that the gravity force can be neglected or, if preferred, P may be considered as the resultant of the gravity force and some other force not shown. The actual distribution of pressure over the area of contact between the block and the plane will depend upon the point of application of the force P and



(a) (b)

Fig. 2.70

Solution: When conditions are such that sliding of the block impends, it will be in equilibrium under the action of three forces: the gravity force W, the pull $P_{\min}$, and the reaction R which is the resultant of the distributed pressure exerted on the block by the floor [Fig. 2.73(a)]. These three forces must meet in one point and must build a closed triangle when geometrically added. To construct this triangle of forces, we begin with the known vector W, and from the end of this we draw the line that makes the angle of friction φ with it and that is the known limiting direction of the reaction R when motion impends. It is now evident that the shortest vector $P_{\min}$ which will make the closing side of the triangle is one at right angles to the reaction R. Thus we conclude that the least force that will cause sliding of the block to impend will be one making the angle of friction φ with the plane of the floor. That is, $\alpha = \varphi$.

From the triangle of forces [Fig. 2.73(b)], the magnitude of this least force with which the block can be made to slide is found to be

$$P_{\min} = W \sin \varphi \tag{g}$$

Comparing Eqs (g) and (f), it is seen that, for cases in which the coefficient of friction is fairly large, considerable effort in sliding a heavy block over a rough surface will be saved by pulling along a line that makes the angle of friction with the plane of sliding. For example, in the case of stone sliding on concrete (assuming $\mu = 0.6$) the least force $P_{\min}$ as given by Eq. (g), will be only 86 per cent of the horizontally applied force P as given by Eq. (f).

4. To raise a heavy stone block weighing 8.9 kN, the arrangement shown in Fig. 2.74(a) is used. What horizontal force P will it be necessary to apply to

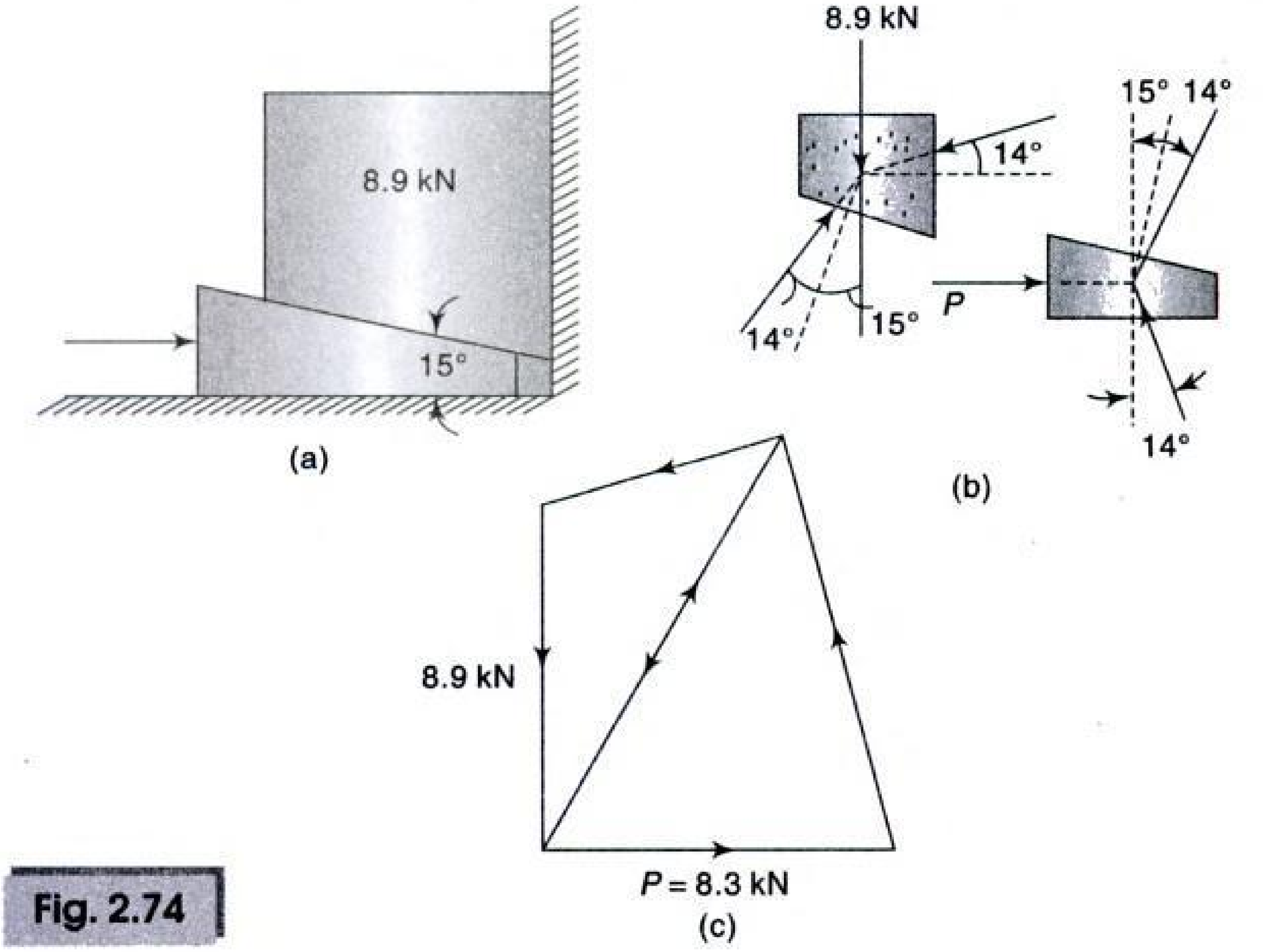


Fig. 2.74

6. A body of weight W is placed on an inclined plane. The angle made by the inclined plane with the horizontal, when the body is on the point of moving down is called
   (a) Angle of inclination (b) Angle of repose
   (c) Angle of friction (d) Angle of limiting friction
   [*Ans.* (b)]
7. A body of weight Q is placed on an inclined rough plane. The inclination of the plane with the horizontal is less than the angle of friction. The body will
   (a) be in motion (b) move downwards
   (c) be in equilibrium (d) move upwards
   [*Ans.* (c)]
8. On a ladder resting on a smooth ground and leaning against a rough vertical wall, the force of friction acts
   (a) towards the wall at its upper end (b) away from the wall at its upper end
   (c) upwards at its upper end (d) downwards at its upper end
   [*Ans.* (c)]
9. If a ladder is not in equilibrium against a smooth vertical wall, then it can be made in equilibrium by
   (a) increasing the angle of inclination with horizontal
   (b) increasing the area of the ladder
   (c) decreasing the angle of inclination with horizontal
   (d) decreasing the area of the ladder [*Ans.* (a)]

PROBLEM SET 2.7

1. What must be the angle α between the plane faces of a steel wedge used for splitting logs if there is to be no danger of the wedge slipping out after each blow of the sledge? (*Ans.* $\alpha \leq 2\varphi$)
2. A flat stone slab rests on an inclined skidway that makes an angle α with the horizontal. What is the condition of equilibrium if the angle of friction is φ? (*Ans.* $\alpha \leq 2\varphi$)
3. What is the necessary coefficient of friction between tires and roadway to enable the four-wheel-drive automobile in Fig. A to climb a 30 per cent grade? (*Ans.* $\mu \geq 0.3$)

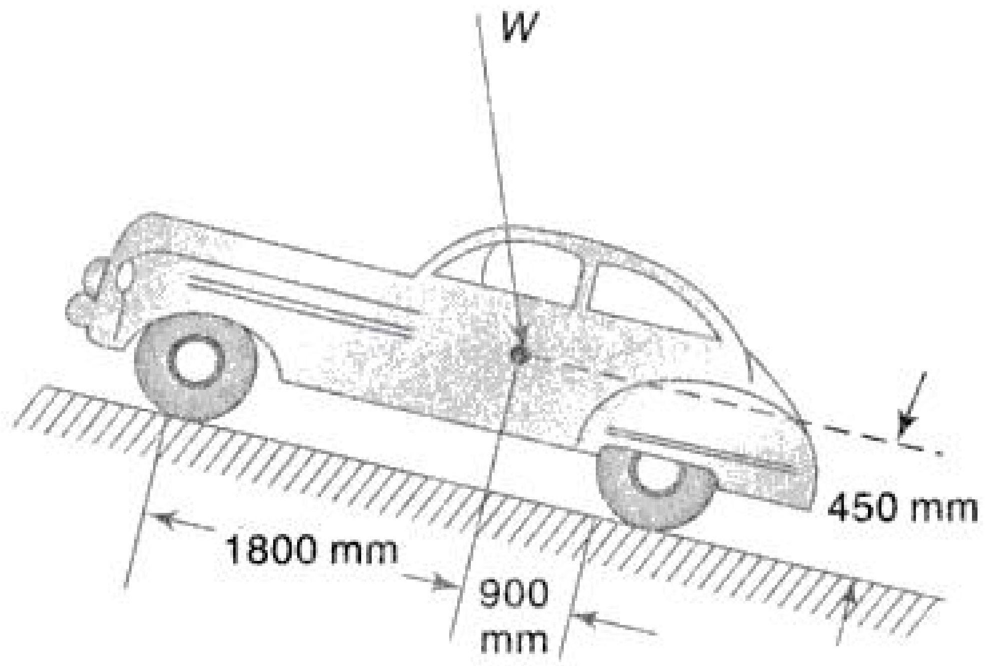


Fig. A

4. A heavy rotating drum of radius r is supported in bearings at C and is braked by the device shown in Fig. B. Calculate the braking moment M_c with respect to point C if the coefficient of kinetic friction between drum and brake shoe is μ. (*Ans.* $\mu Plr/a$)
5. To determine experimentally the coefficient of friction for steel on steel, flat plates of negligible weight compared with the large top weight W, are stacked on a hori-

cient of friction μ has the same value for all surfaces of contact, determine the necessary condition under which the large roller can be pulled over the small one.

(*Ans.* $\mu \geq \sqrt{d/D}$)

*17. A solid right circular cone of altitude $h = 304.6$ mm and radius of base $r = 76.2$ mm has its center of gravity C on its geometric axis at the distance $h/4 = 76.2$ mm above the base. This cone rests on an inclined plane AB, which makes an angle of 30° with the horizontal and for which the coefficient of friction is $\mu = 0.5$ (Fig. O). A horizontal force P is applied to the vertex O of the cone and acts in the vertical plane of the figure as shown. Find the maximum and minimum values of P consistent with equilibrium of the cone if the weight $W = 44.5$ N.

(*Ans.* $P_{\max} = 20.52$ N; $P_{\min} = 2.67$ N)

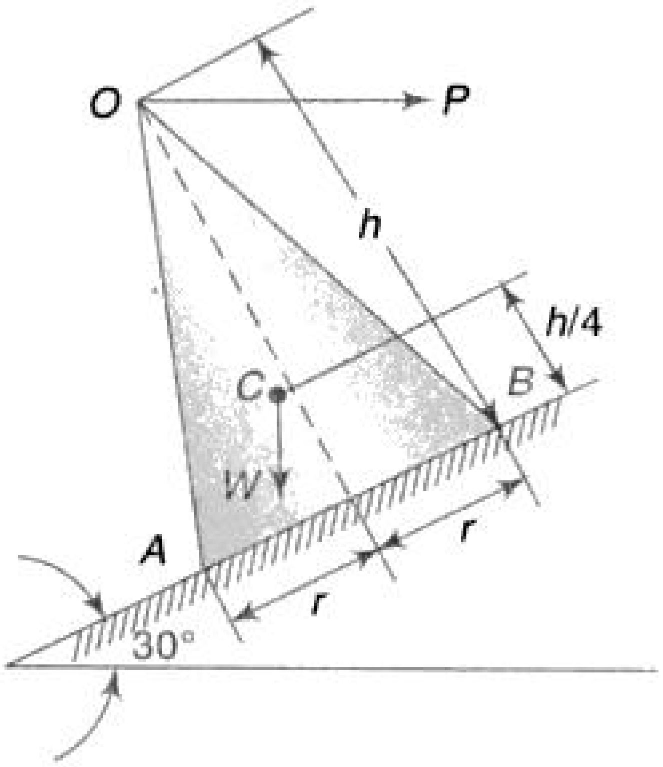


Fig. O

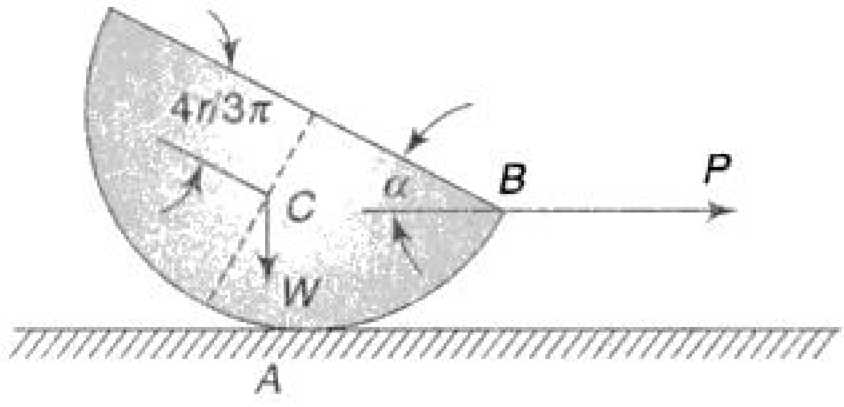


Fig. P

*18. A short semicircular right cylinder of radius r and weight W rests on a horizontal surface and is pulled at right angles to its geometric axis by a horizontal force P applied at right angles to its geometric axis by a horizontal force P applied at the middle B of the front edge (Fig. P). Find the angle α that the flat face will make with the horizontal plane just before sliding begins if the coefficient of friction at the line of contact A is μ. The gravity force W must be considered as acting at the center of gravity C as shown in the figure. (*Ans.* $\sin \alpha = 3\mu\pi/(4 + 3\mu\pi)$)

$$Q \cdot CB' - P \cdot CA' = 0$$

from which

$$\frac{CB'}{CA'} = \frac{P}{Q} \tag{h}$$

Thus again the distances of the components from the line of action of the resultant are inversely proportional to their magnitudes, but the line of action of the resultant lies outside the space between the components on the side of the larger force.

Two Equal Parallel Forces Acting in Opposite Directions

A system of two equal parallel forces acting in opposite directions cannot be reduced to one resultant force. This can readily be seen from Fig. 3.3. If the forces P and Q are equal in magnitude, the two parallelograms APP_1S and BQQ_1S will be equal and corresponding sides will be parallel. Hence P_1 and Q_1 will still be equal and oppositely directed parallel forces, and we never obtain a point of intersection C. Thus, we cannot reduce two equal and opposite but noncollinear forces to any simpler system. Two such forces are called a *couple*, the plane in which they act is called *the plane of the couple*, and the distance between their lines of action is called the *arm of the couple*.

The algebraic sum of the moments of the two forces of a couple is independent of the position, in the plane of the couple, of the moment center and is always equal to the product of the magnitude of either force and the arm of the couple. Consider, for example, the couple PP with arm AB of length a in Fig. 3.4, and let O be any arbitrary moment center. Then the algebraic sum of moments is

$$P \cdot OD - P \cdot OC = P(OD - OC) = Pa$$

The same result will be obtained if O lies between the lines of action of the forces. The moment Pa is called the *moment of the couple*. It is positive when the couple tends to produce counter-clockwise rotation as shown, and it is negative for clockwise rotation.

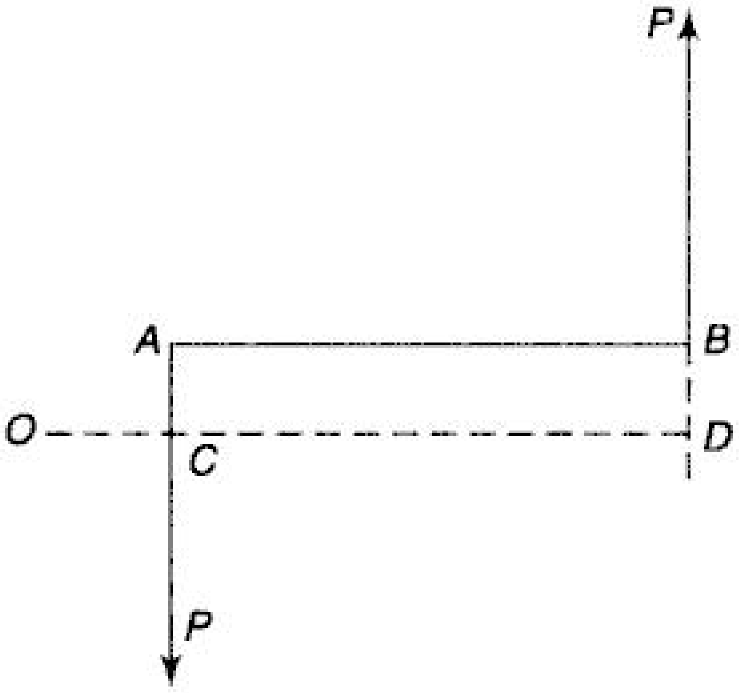


Fig. 3.4

The action of a couple on a rigid body will not be changed if its arm is turned in the plane of the couple through any angle α about one of its ends. To prove this statement, let us consider a couple PP with the arm AB, as shown in Fig. 3.5. We take under any desired angle α with the arm AB a straight-line segment AC equal in length to the arm AB of the given couple. At each of the ends A and C of this line we apply two equal and opposite forces Q and Q' perpendicular to AC and equal in magnitude to P. From the principle of superposition it follows that the addition of these forces which are in equilibrium does not change the action of the given couple. The forces Q together with the forces P give the resultant SR,

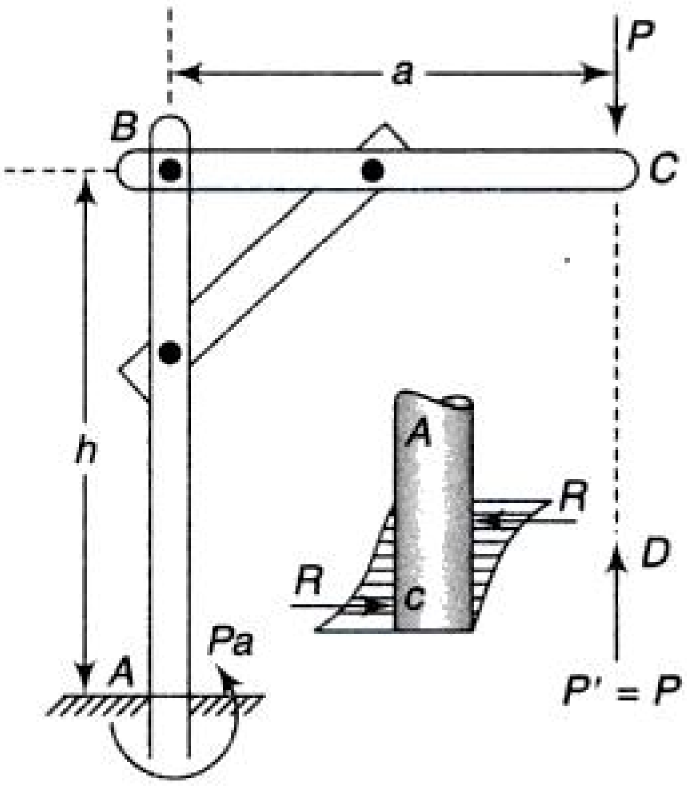
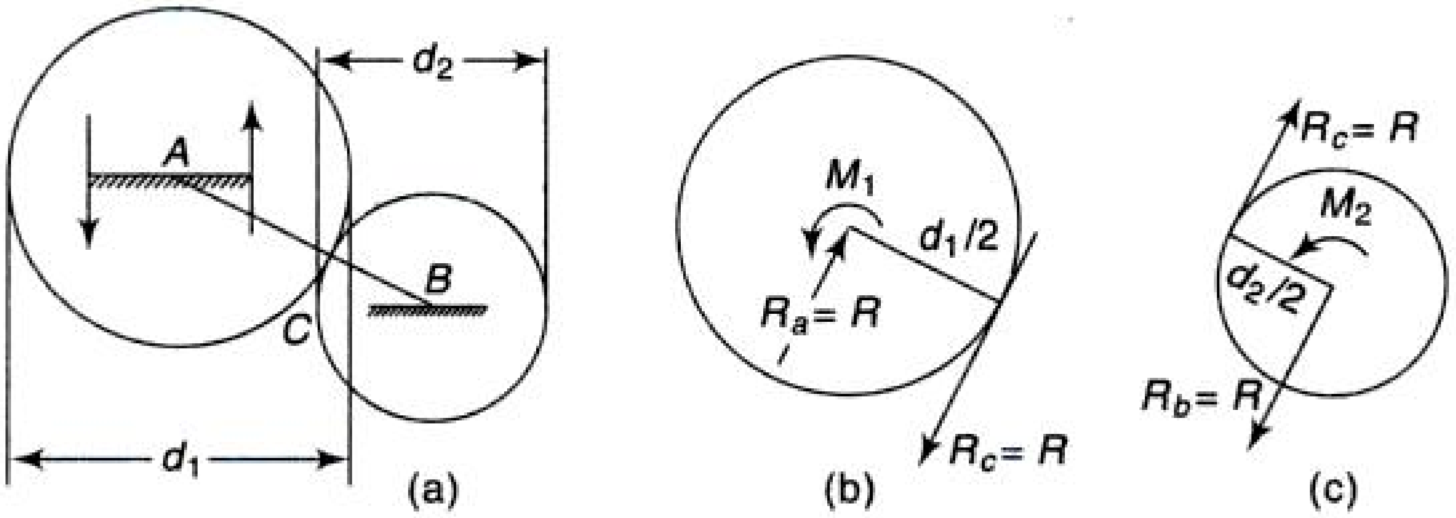


Fig. 3.10

Solution: Clearly, the reaction at *A* must be statically equivalent to the equilibrant *P′* of the active force *P*, that is, a vertical upward force acting along the line *DC*. To represent this force as a reaction at *A*, we resolve it into an equal parallel force *P* at *A* and a couple of moment *Pa* acting as shown in the figure. To understand physically just how the foundation exerts this couple on the bottom of the mast, we must consider the detail of the bottom of the mast as shown in Fig. 3.10. In trying to tip over to the right, the mast presses against the walls of the hole in which it is seated and, of course, equal and opposite reactive pressure will be exerted on the mast. This distribution of pressure will be something like that indicated in the figure and is sensibly equivalent to a couple *RR* acting as shown. We see that this kind of constraint, a so-called *built-in end*, is considerably more complex than any we have previously discussed. This example illustrates one of the many useful applications of the resolution of a force into a force and a couple.

3. Two gears having pitch diameters d_1 and d_2 are connected as shown in Fig. 3.11(a). If a couple of moment M_1 is applied to the upper gear as shown, what is the moment M_2 of the couple that must be applied to the lower gear to maintain equilibrium?



(a) (b) (c)

Fig. 3.11

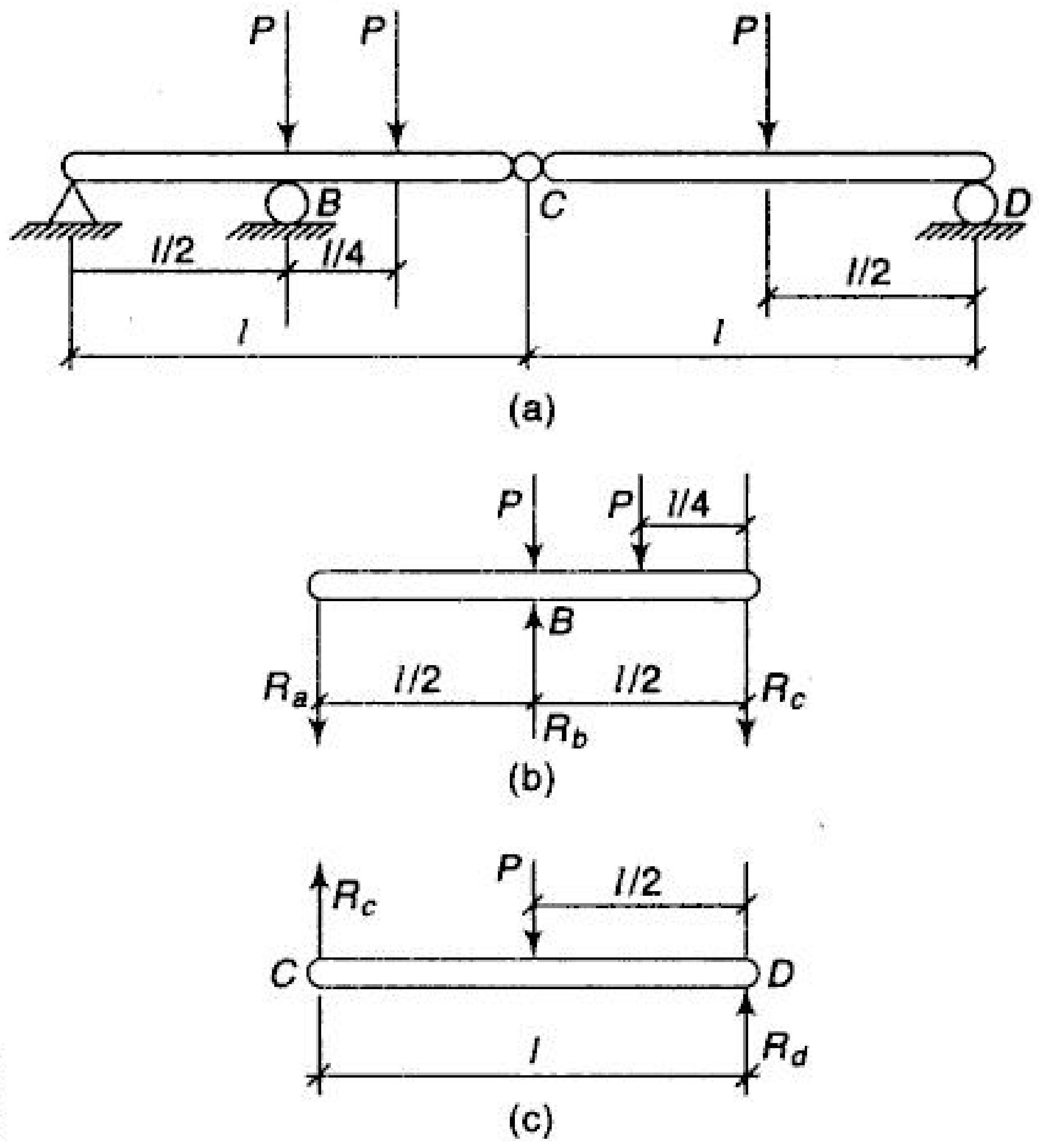


Fig. 3.21

Thus acting on the beam *CD*, we have the vertical forces R_c, *P* and R_d in equilibrium. The simplest way of calculating the reactions R_c and R_d is by using Eq. (12). Taking moments, first with respect to point *C* and then with respect to point *B*, we obtain the equations.

$$R_d l - \frac{Pl}{2} = 0 \quad -R_c l + \frac{Pl}{2} = 0$$

from which

$$R_d = \frac{P}{2} \quad \text{and } R_c = \frac{P}{2}$$

Now consider the free body of beam *ABC*. The loads acting on this beam are two active forces *P*, the reaction R_b from the roller support at *B* and the reaction R_c from the hinge *C* acting downward. We can reduce R_c and *P* into a resultant force *R* vertical. Since *R* and R_b are parallel forces, so the reaction R_a at *A* must be vertical for the system to be in equilibrium as discussed for the previous beam. Taking moments, first with respect to point *A* and then with respect to point *B*, we obtain the equations

$$R_b \frac{l}{2} - \frac{Pl}{2} - P\left(\frac{3l}{4}\right) - R_c l = 0$$

$$R_b \frac{l}{2} - \frac{Pl}{4} - \frac{R_c l}{2} = 0$$

Substituting the value of R_c in the above equations, we get the reactions R_a and R_b as shown below:

$$R_a = P \text{ and } R_b = 3.5\, P$$

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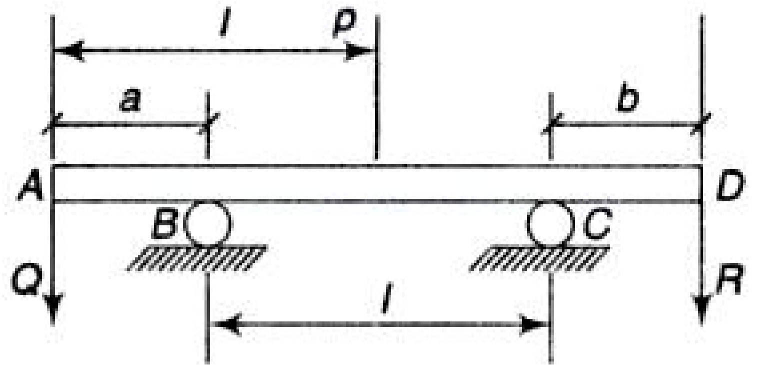


Fig. H

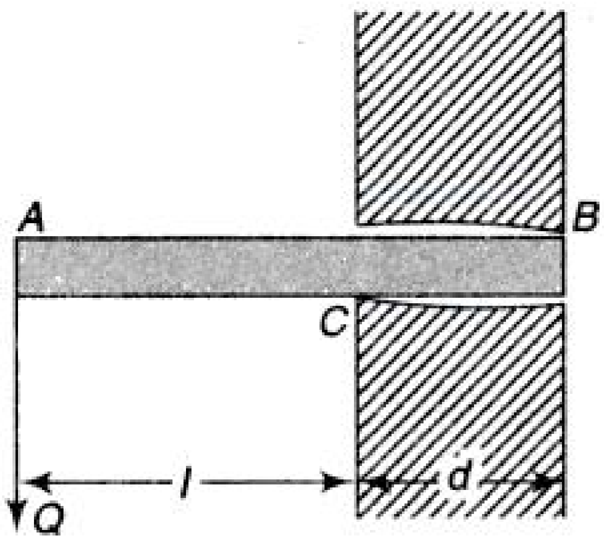


Fig. I

10. Determine the reactions at A for the cantilever beam AB subjected to the loads as shown in Fig. J. Numerical data are given: $P = 1500$ N, $Q = 1000$ N, $R = 1500$ N and

$$a = b = c = \frac{l}{3}, l = 3 \text{ m}.$$

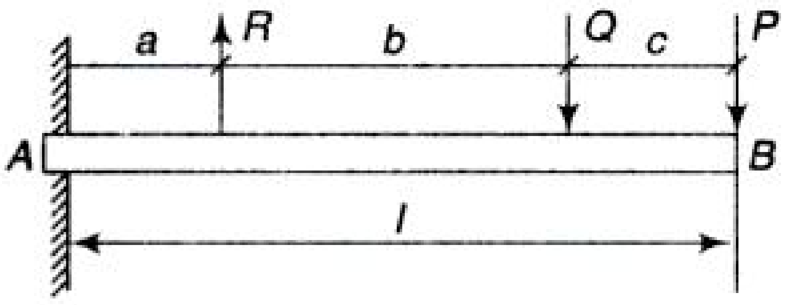


Fig. J

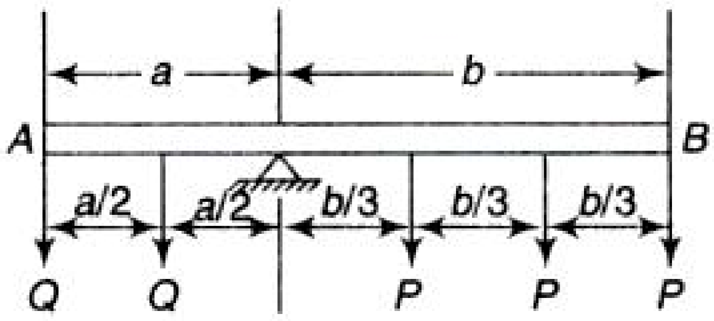
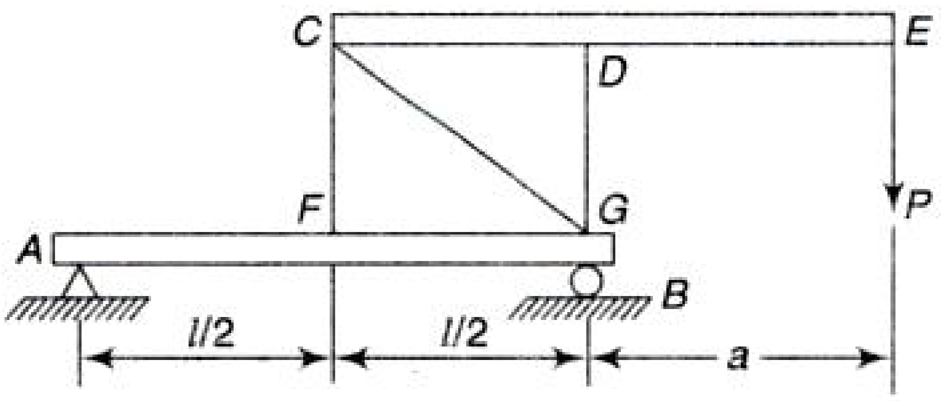


Fig. K

11. Along a lever AB loads Q and P are distributed, as shown in Fig. K. If $Q = 2P$ and the weight of the lever is negligible, determine the ratio $a : b$ of the arms of the lever if it is in equilibrium. (*Ans.* $a : b = 2 : 3$)
12. The beam CE in Fig. L is supported on the beam AB by the three bars CF, DG and CG, as shown. Find the reactions that will be produced at the points of support A and B of the lower beam due to the action of a load P applied at the free end E of the upper beam if the span $l = 3.6$ m and $a = 1.2$ m. (*Ans.* up; down)

Fig. L



13. Find the load P required to maintain the system of levers in equilibrium with AB in horizontal position, if $Q = 9.6$ kN (Fig. M). (*Ans.* $P = 200$ N)
14. Two identical prismatic bars AB and CD are welded together in the form of a rigid T and suspended in a vertical plane as shown in Fig. N. Calculate the angle α that the bar CD will make with the vertical when a vertical load $P = 44.5$ N is applied at B. The weight of each bar is $Q = 22.25$ N as shown. (*Ans.* $a = 15°57'$)

$$x_c = \frac{\Sigma(F_i x_i)}{\Sigma F_i} \tag{13a}$$

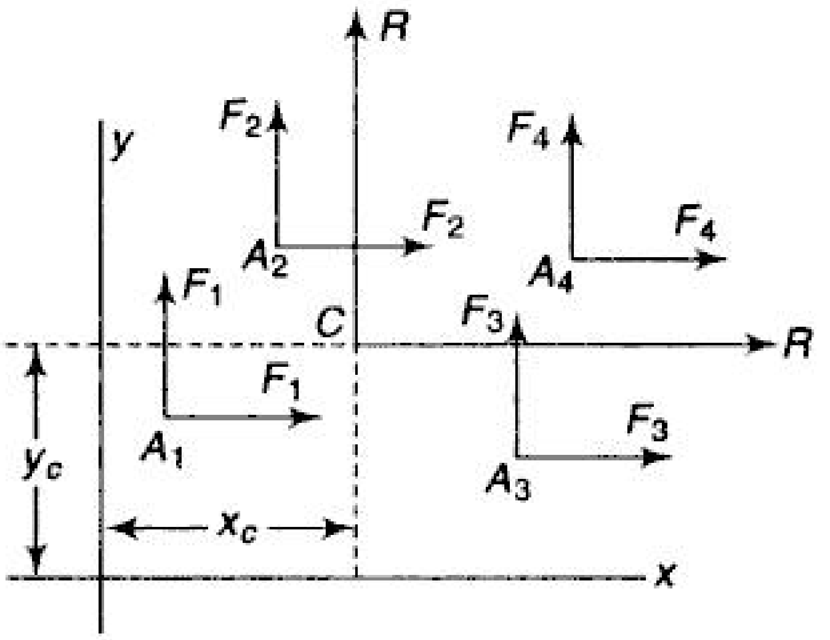


Fig. 3.24

Now let the forces all be rotated in the plane of the figure until they act parallel to the x-axis. In this case the arm y_c of the resultant may be found by again using the second of Eq. (9), and we obtain

$$y_c = \frac{\Sigma(F_i y_i)}{\Sigma F_i} \tag{13b}$$

We have already seen that the center of parallel forces of any number of forces applied at a given system of points is independent of the direction in which the forces act. Hence, we conclude that the moment arms x_c and y_c as defined by Eq. (13) represent the coordinates of this point.

The center of parallel forces for a given system of forces $F_1, F_2, \ldots, F_n$ applied at given points $A_1, A_2, \ldots, A_n$ in one plane will not be changed if the magnitudes of the forces are all multiplied by the same constant factor. This statement follows at once from the form of Eq. (13) from which we see that such a factor will appear n times in both the numerator and the denominator of either of these expressions and will therefore cancel out. Thus we conclude that the center of parallel forces for any given system of forces applied at a given system of points in one plane depends only upon the positions of the points and upon the relative magnitudes of the forces.

Center of Gravity

The center of gravity of a body is that point through which the resultant of the distributed gravity force passes regardless of the orientation of the body in space. From this definition it follows that the center of gravity of a rigid body is the center of parallel gravity forces acting on the various particles of the body. Since gravity forces always act vertically downward, it is evident that some rotation of a body through an angle α is equivalent to a corresponding rotation through the same angle of all the gravity forces about their points of application as discussed above.

All physical bodies are, of course, three-dimensional as a consequence of which the gravity forces acting on the various particles of the body represent a system of

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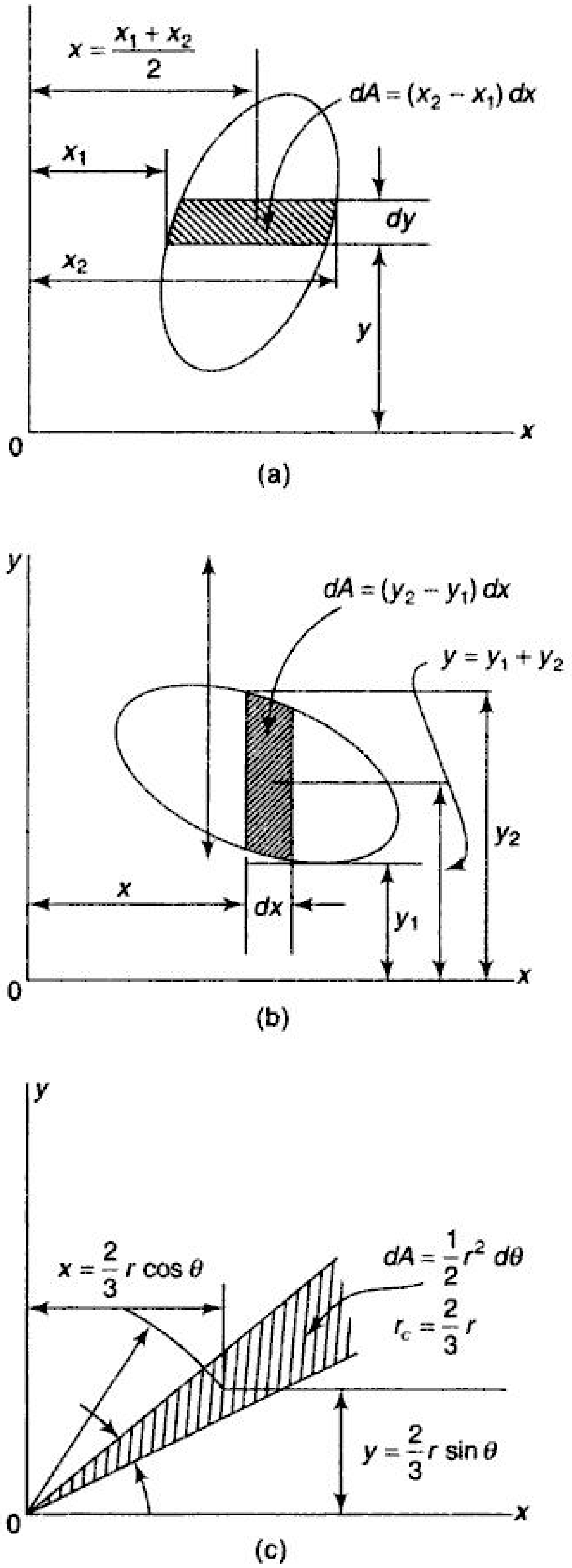

$x = \frac{x_1 + x_2}{2}$
$dA = (x_2 - x_1)\,dx$
x_1
dy
x_2
y
0
x
(a)
y
$dA = (y_2 - y_1)\,dx$
$y = y_1 + y_2$
y_2
x
dx
y_1
0
x
(b)
y
$x = \frac{2}{3} r \cos\theta$
$dA = \frac{1}{2} r^2\,d\theta$
$r_c = \frac{2}{3} r$
$y = \frac{2}{3} r \sin\theta$
0
x
(c)

Fig. 3.27

The expressions for the differential length *dL* for rectangular and polar coordinates are shown in Fig. 3.28.

Sometimes the position of the centroid of a plane figure or curve can be seen by inspection. For example, if a figure has two axes of symmetry, its centroid lies at their intersection. This statement follows at once from the form of expressions (16) and (17).

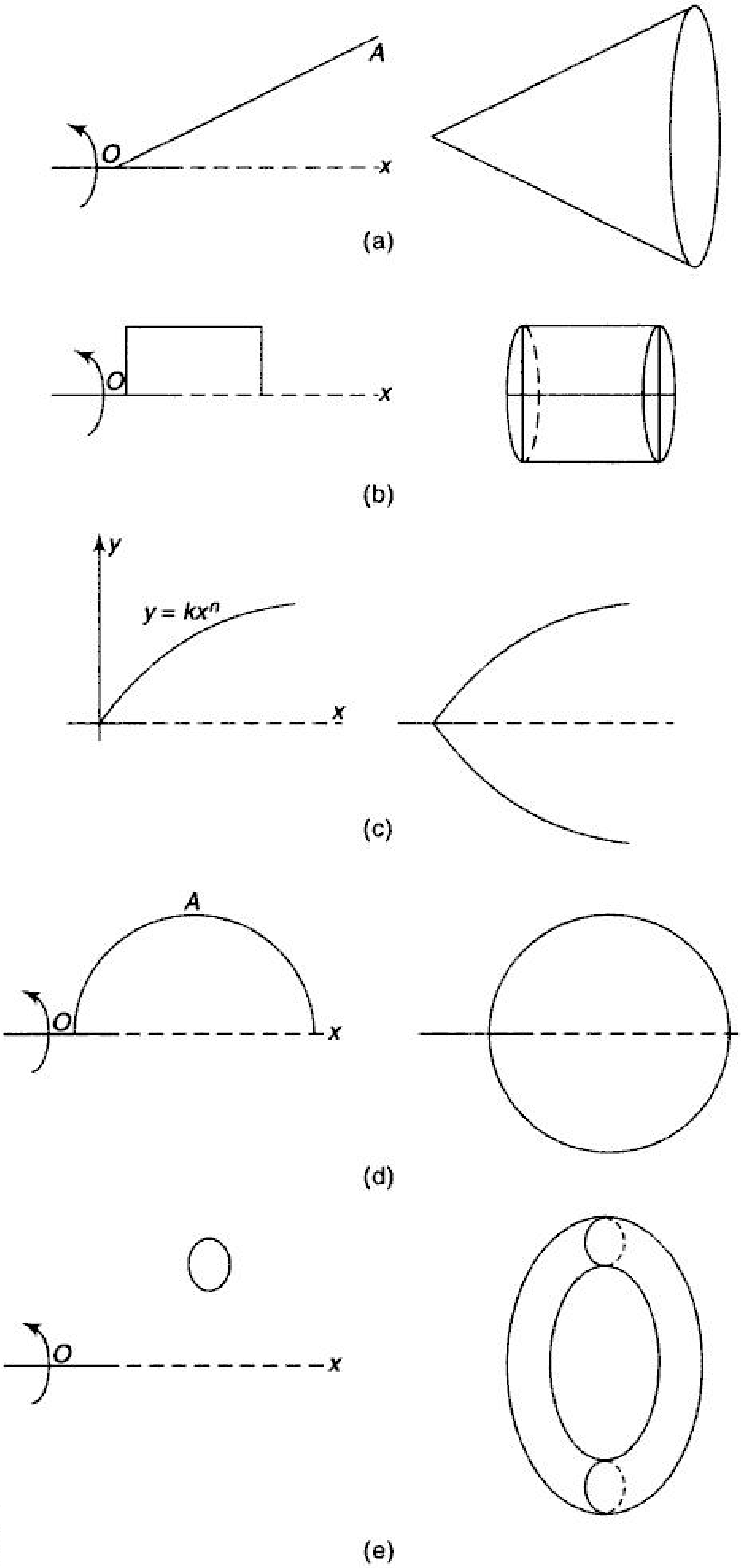
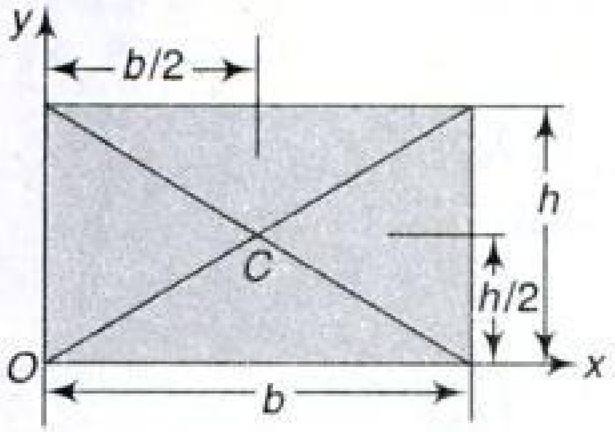
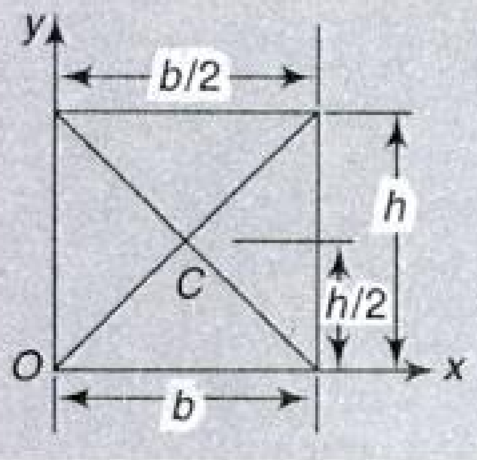
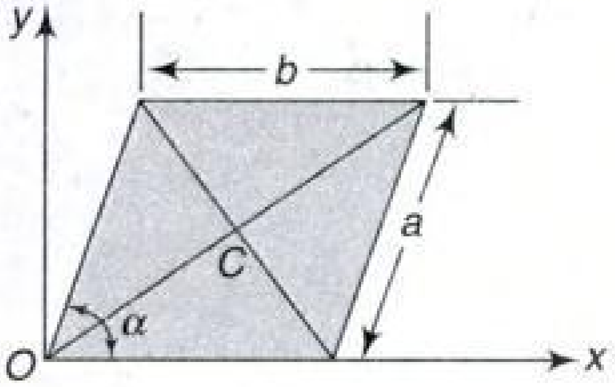
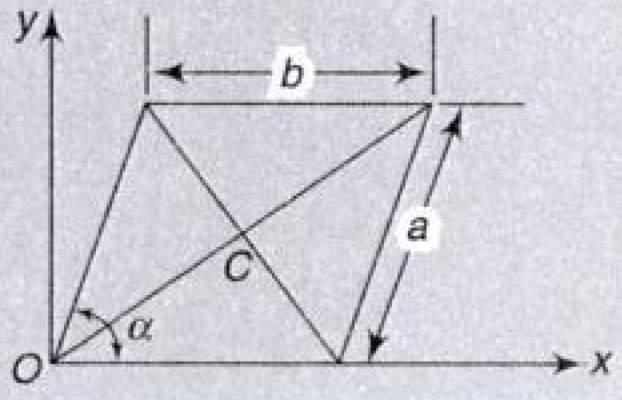
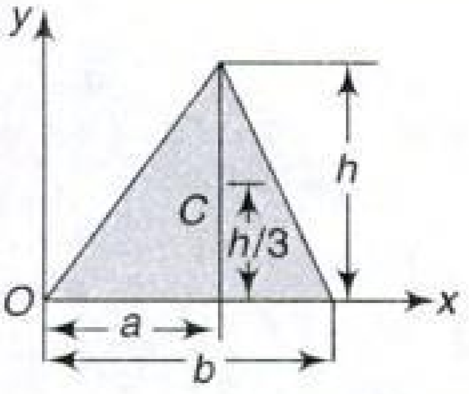

A
O
x
(a)
O
x
(b)
y
$y = kx^n$
x
(c)
A
O
x
(d)
O
x
(e)

Fig. 3.36

Various Centroids of Plane Areas

Description	*Shape*	*Area*	x_c	y_c
Rectangle	 y b/2 h C h/2 O x b	bh	$b/2$	$h/2$
Square $(h = b = a)$	 y b/2 h C h/2 O x b	a^2	$a/2$	$a/2$
Parallelogram		$ab \sin \alpha$	$\dfrac{b + a\cos\alpha}{2}$	$\dfrac{a\sin\alpha}{2}$
Rectangle $(a = \pi/2)$		ab	$b/2$	$a/2$
Triangle		$1/2bh$	$1/3(a + b)$	$h/3$

(*Contd.*)

L of the curve and the distance travelled by its centroid. 2. The volume of the solid generated by rotating any plane figure about a nonintersecting axis in its plane is equal to the product of the area A of the figure and the distance travelled by its centroid.

- A surface of revolution is a surface generated curve by rotating a plane curve about a fixed axis.
- A body of revolution is a body generated by rotating a plane area about a fixed axis.
- The theorems of Pappus are very useful in calculating the surface areas and volumes of various bodies of revolution encountered in engineering, particularly in machine design.
- The theorems of Pappus can also be used to determine the centroid of a plane curve when the area of the surface generated by the curve is known or to determine the centroid of a plane area when the volume of the body generated by the area is known.

Important Formulae

1. The center of parallel forces for a given system of forces applied at the given point in one plane can be obtained by the following equations:

$$x_c = \frac{\Sigma(F_i\, y_i)}{\Sigma F_i}, \qquad y_c = \frac{\Sigma(F_i\, y_i)}{\Sigma F_i}$$

2. Analytically, the position of the centroid of area of a plane figure may be defined by the formulae and the summations are understood to include all elements of area within the boundary of the figure.

$$x_c = \frac{\Sigma(\Delta A_i\, x_i)}{\Sigma \Delta A_i}, \qquad y_c = \frac{\Sigma(\Delta A_i\, y_i)}{\Sigma \Delta A_i}$$

3. The position of the centroid of length of a plane curve may be defined by the formulas

$$x_c = \frac{\Sigma(\Delta L_i\, x_i)}{\Sigma \Delta L_i}, \qquad y_c = \frac{\Sigma(\Delta L_i\, y_i)}{\Sigma \Delta L_i}$$

   where L_i denotes the length of an element and the x_i, y_i coordinates of its midpoint.

4. The coordinates of the centroid of any plane figure or curve can be calculated, provided the integrals appearing therein can be evaluated.

$$x_c = \frac{\int x\,dA}{\int dA}, \qquad y_c = \frac{\int y\,dA}{\int dA}$$

   for the case of area and

$$x_c = \frac{\int x\,dL}{\int dL}, \qquad y_c = \frac{\int y\,dL}{\int dL}$$

   for the case of length.

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Solution: Note that the shaded triangles *mpC* and *nqC* are geometrically similar. Hence, we may write

$$\frac{y_c}{h - y_c} = \frac{a/2 + b}{a + b/2}$$

which is readily brought to the form $y_c = \dfrac{a(a + 2b)}{3(a + b)}$

as previously obtained in Example 1 above.

3. Locate the centroid *C* of the shaded area of the figure *BDE* (Fig. 3.51) which is obtained by cutting the quadrant of a circle of radius *a* from a square *OBDE* of the same dimensions.

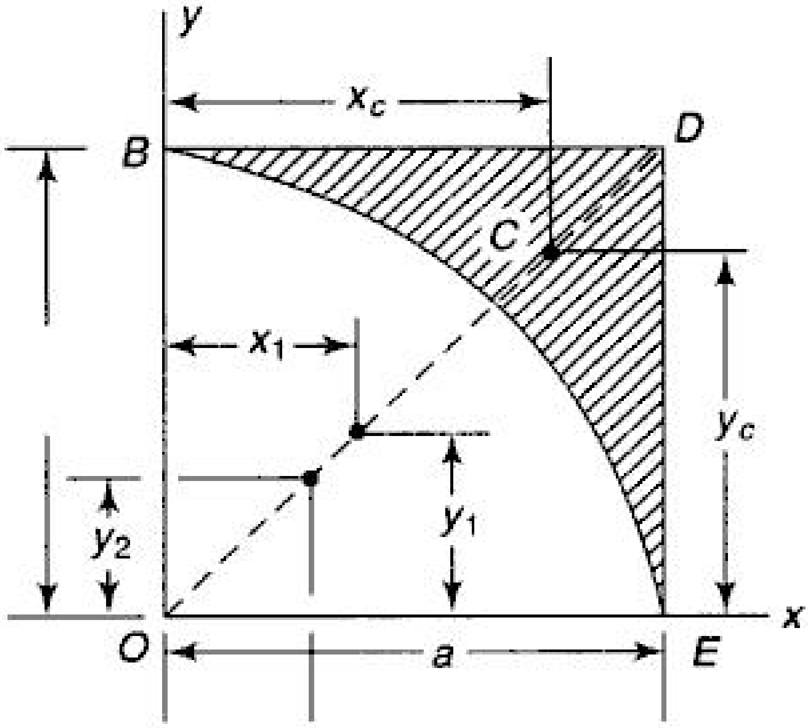


Fig. 3.51

Solution: From symmetry of the figure, it is evident that the desired centroid *C* lies on the diagonal *OD* of the square. Hence, choosing coordinate axes *x* and *y* as shown in the figure, it will only be necessary to determine one coordinate of the centroid, since $x_c = y_c$.

To determine x_c, let us denote by A_1 the area of the square, by x_1 the *x* coordinate of its centroid C_1, and by A_2, x_2 the corresponding quantities for the quadrant of the circle. Then the first of Eq. (14) gives

$$xc = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2}$$

For the given dimensions and remembering that $x_2 = y_2 = 4a/3\pi$ this becomes

$$x_c = \frac{a^2(a/2) - (\pi a^2/4)(4a/3\pi)}{a^2 - \pi a^2/4}$$

from which

$$y_c = x_c = \frac{2a}{3(4 - \pi)} = 0.777a \qquad \text{(e)}$$

Solution: In dealing with this figure, divide the lengths L_1, L_2, L_3, L_4 and L_5 with the centroids C_1, C_2, C_3, C_4 and C_5 as shown. Using the given dimensions, we may fill in the horizontal lines of the table given on next page:

No.	$\Delta A_i\ (mm^2)$	$x_i\ (mm)$	$y_i\ (mm)$	$\Delta A_i x_i\ (mm^2)$	$\Delta A_i y_i\ (mm^2)$
1	78.54	31.83	81.83	2500.00	6427.00
2	78.54	68.17	68.17	5354.07	5354.07
3	50.00	0.0	25.00	0.00	1250.00
4	50.00	100	25.00	500.00	1250.00
5	100	0.0	0.00	0.00	0.00
Σ	6696			12854.07	14281.07

Equation (15) gives,

$$y_c = \frac{12854.07}{357.08} = 36 \text{ mm}$$

$$y_c = \frac{14281.07}{357.08} = 40 \text{ mm}$$

8. A homogeneous prismatic bar *ABC* of negligible diameter, is bent into a straight section and a semicircular arc as shown in Fig. 3.56. It is attached to a hinge at *A*. Determine the value of α for which the bar is in equilibrium for the indicated position.

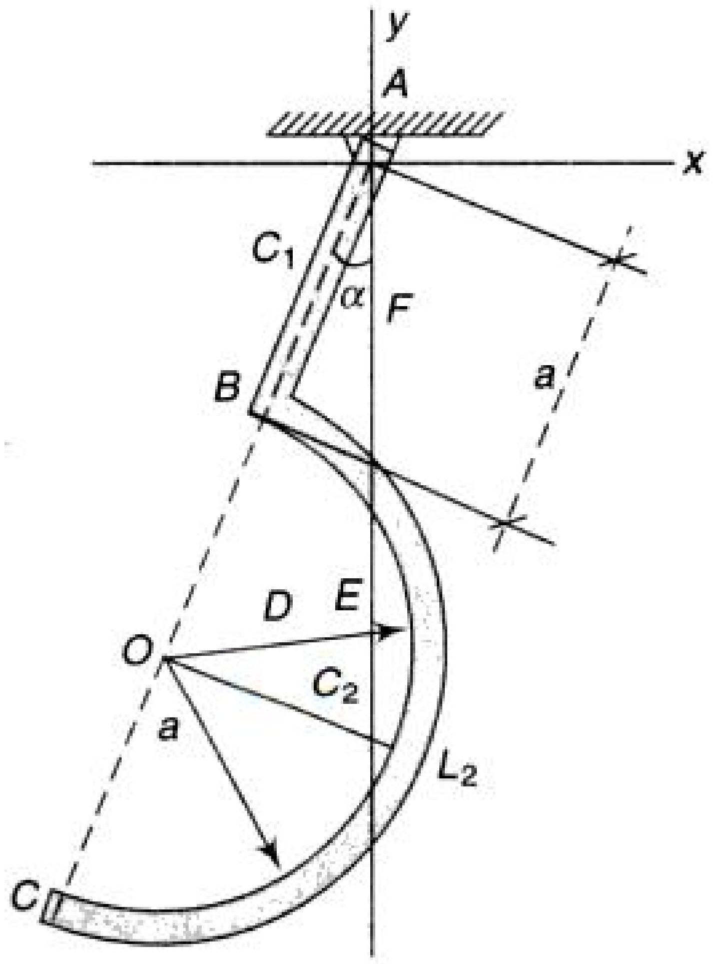


Fig. 3.56

Solution: Since *ABC* is formed of homogeneous wire, its center of gravity coincides with the centroid of the corresponding line. For the body to be in equilibrium, the center of gravity should be on the y-axis, i.e., $x_c = 0$. Dividing the lengths L_1 and L_2 with the centroids C_1 and C_2 as shown. Now with the help of list of Eq. (15),

$$x_c = \frac{L_1 x_1 + L_2 x_2}{L_1 + L_2}$$

Eq. (12) of Section 3.2. Taking moments with respect to points A and B, these equations become

$$-p_b h + Q\frac{h}{3}h = p_a h - Q\frac{h}{3} = 0$$

from which we find $p_a = Q/3$ and $p_b = 2/3Q$. Substituting for Q its value from Eq. (e), we find

$$p_a = \frac{wh}{2}h \quad \text{and} \quad p_b = \frac{wh^2}{2} \tag{f}$$

2. One end of a cantilever beam AC is built into a wall of thickness a as shown in Fig. 3.62. Owing to the action of a load P applied at the free and C, distributed reactions, as represented by the load diagrams AaB' and $A'Bb$, are produced. Find the maximum intensities q_a and q_b of these reactions. Neglect the weight of the beam, and assume all forces to act in its vertical axial plane of symmetry.

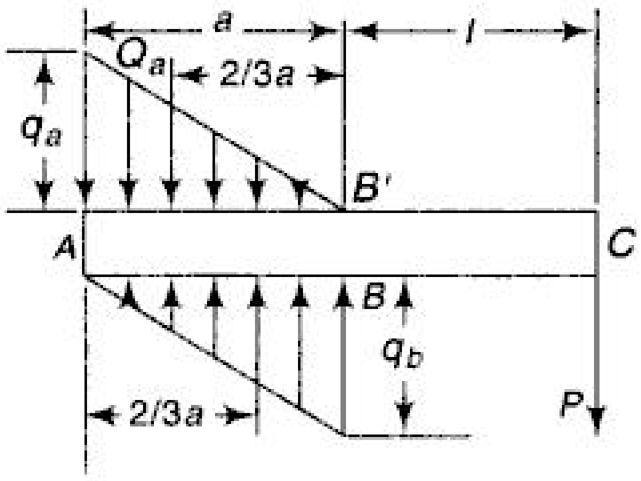


Fig. 3.62

Solution: Since we assume a linear variation in the intensities of pressure along the lines AB' and $A'B$, it follows that the lines of action of the resultants, Q_a and Q_b are as shown in the figure. Replacing the distributed reactions by these resultants, we obtain a system of three parallel forces in a plane, which are in equilibrium. Hence, taking moments of the three forces p, Q_a and Q_b with respect to points D and E and using Eq. (12), we obtain

$$Q_b\frac{a}{3} - P\left(l + \frac{2}{3}a\right) = 0, \qquad Q_a\frac{a}{3} - P\left(l + \frac{a}{3}\right) = 0$$

form which

$$Q_a = \frac{3p}{a}\left(l + \frac{a}{3}\right) \quad \text{and} \quad Q_b = \frac{3p}{a}\left(l + \frac{2}{3}a\right)$$

From the fact that the resultant of a distributed force in a plane is equal to the area of its load diagram, we obtain,

$$Q_a = \frac{q_a a}{2} \quad \text{and} \quad Q_b = \frac{q_b a}{2}$$

from which

$$q_a = \frac{2Qb}{2} \quad \text{and} \quad q_b = \frac{2Qb}{2}$$

4

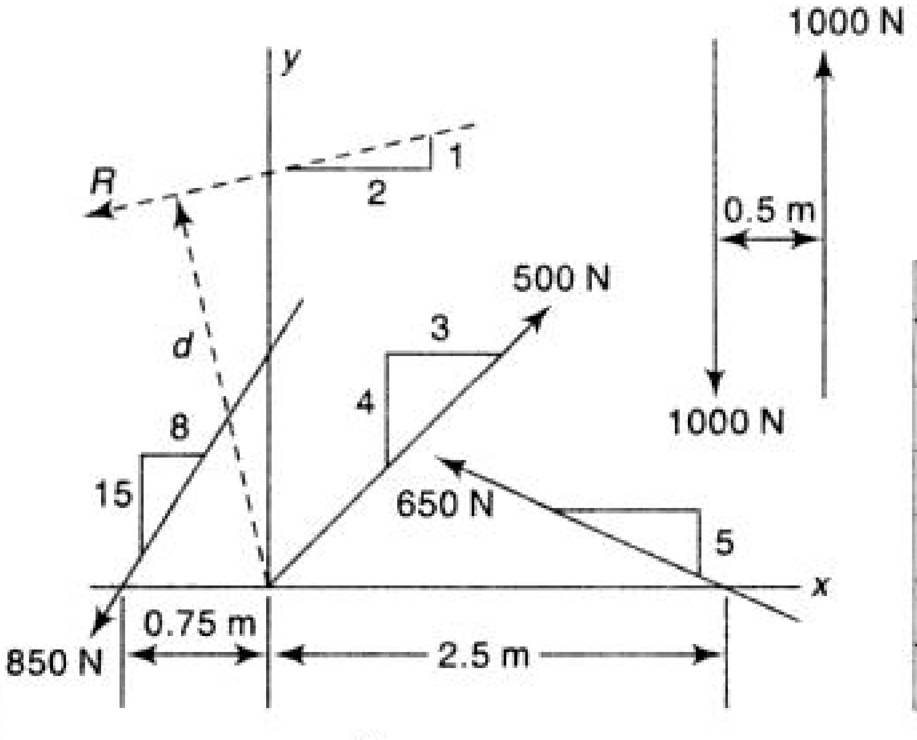
General Case of Forces in a Plane

4.1 COMPOSITION OF FORCES IN A PLANE

If several coplanar forces applied to a body are not parallel and do not intersect in one point, we have the general case of forces in a plane. Let us consider such a system as represented by the forces F_1, ..., F_4, applied, respectively, at points A, B, C and D of the body as shown in Fig. 4.1(a). To find the resultant of these forces, we being with any two forces, say, F_1 and F_2 and determine their resultant R_1 by using the parallelogram law as indicated in the figure. Treating next the forces R_1 and F_3 in the same manner, we find their resultant R_2 which evidently is the resultant of the forces F_1, F_2 and F_3. In the same way again, the resultant R of the forces R_2 and F_4, and consequently of the given system of forces F_1, ..., F_4 is found applied at point G as shown. The point of application of this resultant R may be transmitted to any other point along its line of action if desired. From this discussion, we see that the magnitude and direction of the resultant R are determined by the closing side of the polygon of forces [Fig. 4.1(b)] and are independent of the points of application of the given forces.

If, in a more general case of n forces F_1, ..., F_n in a plane, we find, by successive applications of the parallelogram law that the partial resultant of the first k forces F_1, ..., F_k is parallel to the remaining $n - k$ forces F_{k+1}, ..., F_n. we use the method of addition of parallel forces as discussed in Section 3.2. On the basis of that discussion we conclude that, in general there are three possibilities: (1) The system of n forces in a plane reduces to a resultant force, (2) the system reduces to a resultant couple, or (3) the system is in equilibrium.

To distinguish between these three cases, we begin with the construction of the polygon of forces. If this polygon is not closed [Fig. 4.1(b)], the system reduces to a resultant force, the magnitude and direction of which are given by the closing side of the polygon and the line of action which may be located by using the construction indicated in Fig. 4.1(a) or the method of adding parallel forces.



(a)

F_i	X_i	Y_i	$(M_o)_i$
170	−80	−150	+300
100	+60	+80	0
130	−120	+50	+400
200	0	−200	+300
200	0	+200	
Σ	−140	−20	+1000

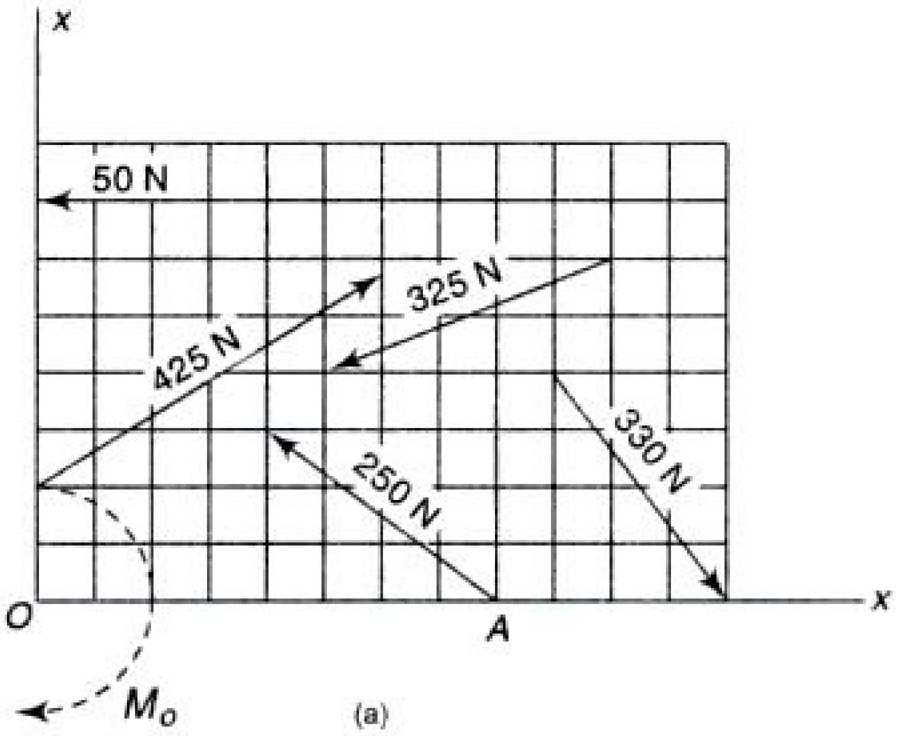
(b)

Fig. 4.4

$$\tan\alpha = \frac{X}{Y} = \frac{-100}{-700} = \frac{1}{7}, \qquad d = \frac{M_0}{R} = \frac{1687.5}{707} = 2.4 \text{ m}$$

Thus the resultant is located as shown in Fig. 4.4.

2. Determine the resultant of the system of coplanar forces shown in Fig. 4.5. Each division of the superimposed grid is 1 cm square.



(a)

F_i	X_i	Y_i	$(M_o)_i$
10	−10.0	0	+70.0
85	+70.7	+47.1	−141.4
65	−60.3	−24.3	+120.6
50	−40.0	+30.0	+240.0
66	+39.6	−52.8	−633.6
Σ	0	0	−344.4

(b)

Fig. 4.5

Solution: We first compute and record the projections X_i and Y_i of each force as shown in Fig. 4.5(b). Summing these projections for all forces, we obtain $X = 0$ and $Y = 0$. Therefore the resultant is not a force.

To see if there is resultant couple, we compute and record the moment $(M_o)_i$ of each force about the origin O. The calculation of these moments is greatly simplified if we resolve each force into its components X_i and Y_i at the point where its line of action intersects either the x- or y-axis, so that one component will pass through O. Take for example, the 250 N force. Resolving it at A into components X_i and Y_i we have

$$(M_o)_i = 200 \times 0 + 150 \times 8 = 1200 \text{ N cm} = 12 \text{ N m}$$

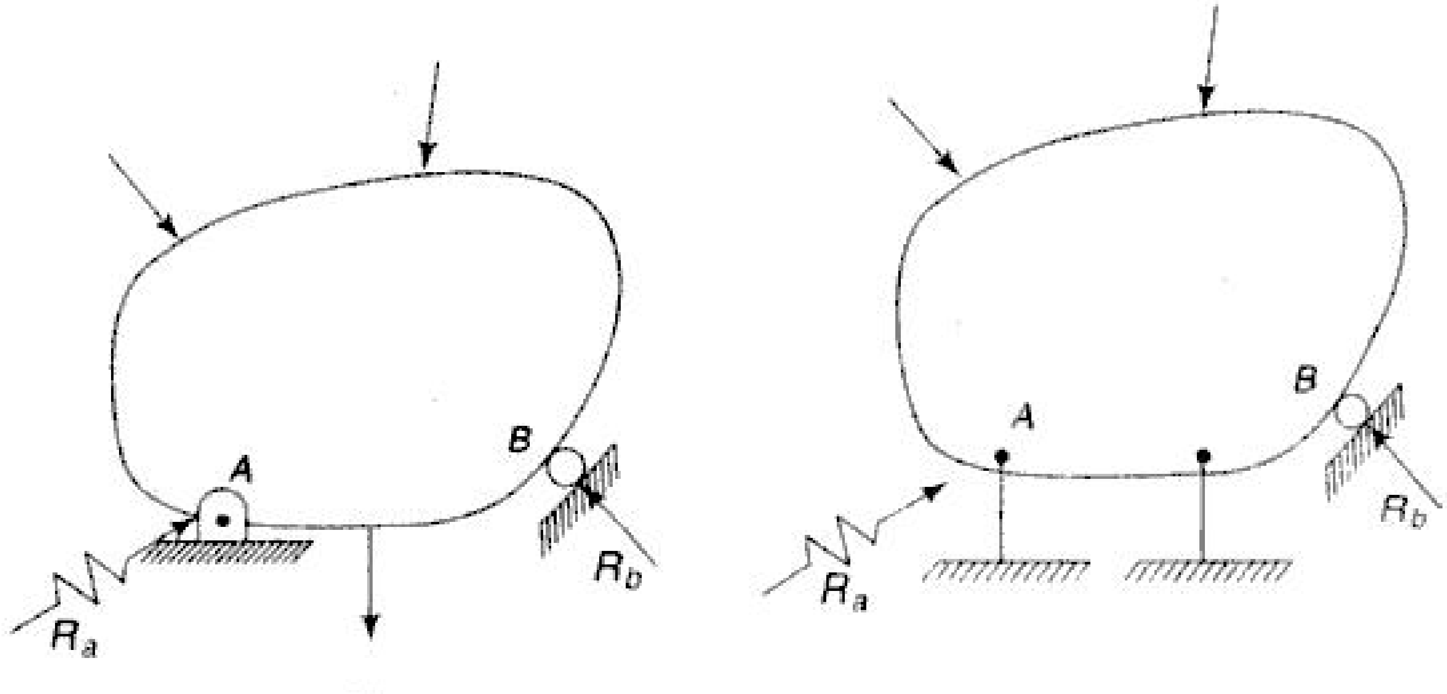
Moments of the forces are calculated in a similar manner. Summing moments in accordance with Eq. (c), we get $M_0 = -\,1732.2$ N cm $= -17.32$ N m. Thus the resultant is a clockwise couple in the plane of action of the forces as shown in Fig. 4.5(a).

resultant is a force acting along the line AB. Finally, if the algebraic sums of the moments of all forces, not only with respect to A and B but also with respect to a third point C, are zero and C is not on the line AB, the possibility of a resultant force falls completely and we must have equilibrium. Expressing these three conditions of equilibrium algebraically, we have

$$\Sigma(M_A)_i = 0, \qquad \Sigma(M_b)_i = 0, \qquad \Sigma(M_c)_i = 0 \tag{19}$$

Reconsidering the set of Eqs (18) or (19), we observe that in both cases, three independent conditions are not only necessary but also sufficient to ensure equilibrium of a system of coplanar force. Naturally, with three equations we can determine only three unknowns. This means that in dealing with constrained bodies where unknown reactions are to be evaluated, we shall not be able to determine the magnitudes of more than three such forces, or possibly the magnitude and direction of one and the magnitude of another. For this reason a system of physical constraints of rigid body in a plane which gives rise to just three unknown is said to be statically determinate.

Common systems of constraints which satisfy the above condition are shown in Fig. 4.6. In Fig. 4.6(a), we have a body supported by a hinge A and a roller B, which completely constrain the body in the plane of the figure. The roller at B determines physically the direction of R_b nut, the direction of R_a remains unknown as indicated by the wavy-line vector. Thus the three unknowns in this case are the magnitude and direction of R_a and the magnitude of R_b. In Fig. 4.6(b), a body is constrained in one plane by three hinged bars which are not parallel and do not intersect in one point[1]. In this case, the lines of action of three reactive forces S_1, S_2 and S_3 (coinciding with the axes of the bars producing them) are known and again we have only three unknown magnitudes to determine.



(a) (b)

Fig. 4.6

Any system of supports of a rigid body in one plane which contains more than three degrees of constraint will set up reactive forces involving more than three

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[1] If the three bars in Fig. 4.6(b) are parallel or intersect in one point, the body will always have some freedom of motion in the plane of the figure and is not adequately constrained, i.e., it will move under the action of applied loads.

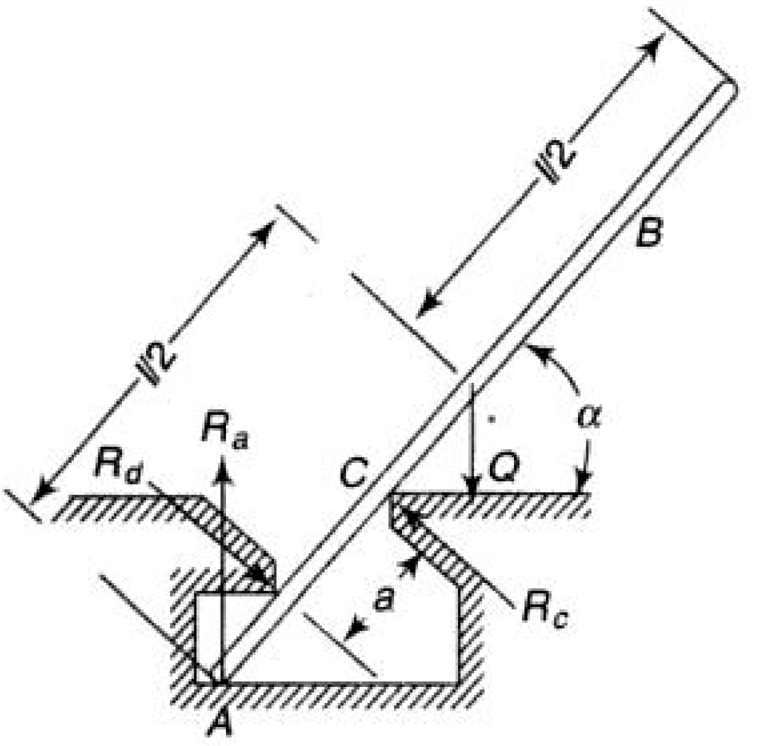


Fig. I

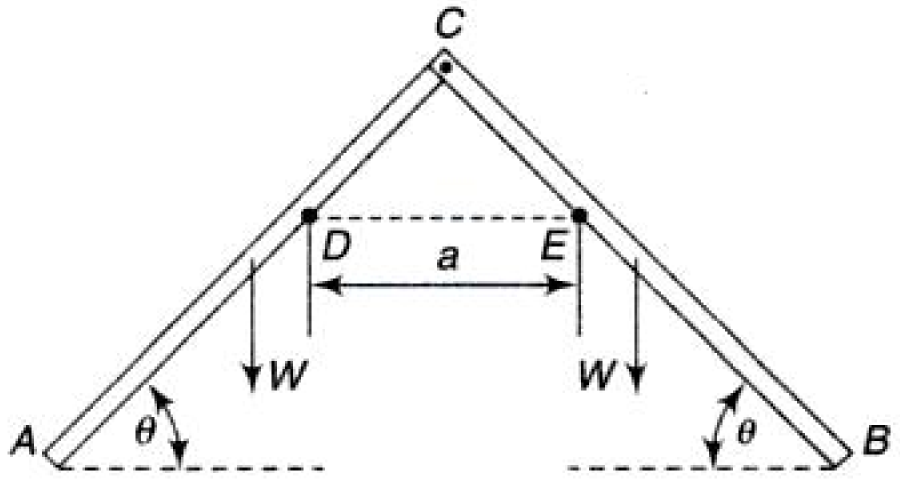


Fig. J

11. A heavy prismatic timber of weight W and length l is supported horizontally between two fixed fulcrums A and B, as shown in Fig. K. If the coefficient of friction between the timber and each fulcrum is μ, find the magnitude of a horizontal force P applied as shown that will cause impending sliding of the timber to the right. The following numerical data are given: $W = 500$ N; $a = 500$ mm, $h = 300$ mm, $l = 1.75$ m, $d = 300$ mm and $\mu = 1/3$. (*Ans.* $P = 250$ N)

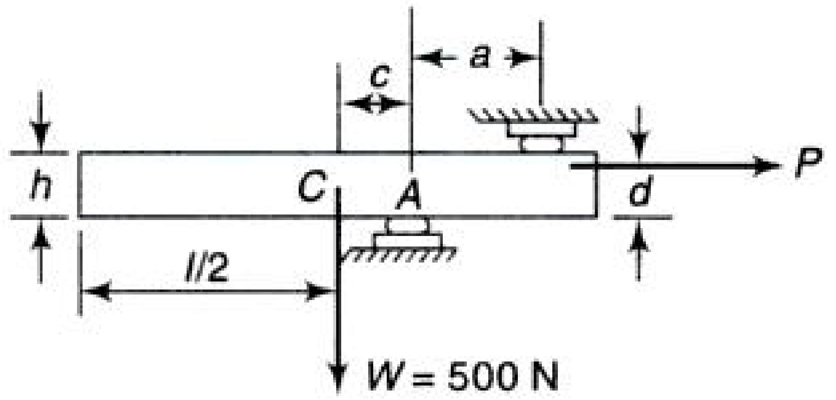


Fig. K

*12. Find the magnitude and direction of the least force P necessary to cause impending sliding of the timber in Fig. K. Compare this result with the value of the corresponding horizontal force as obtained in the preceding problem.

(*Ans.* $P_{\min} = 226$ N, inclined downward by 25°17 to the horizontal)

*13. A heavy prismatic timber of weight W is supported in vertical plane as shown in Fig. L. If the coefficient of friction between the timber and each of the supports A and B is $\mu = 1/3$ and the dimensions are as shown, find the maximum value of the angle α consistent with equilibrium. (*Ans.* $\alpha \leq 43°$)

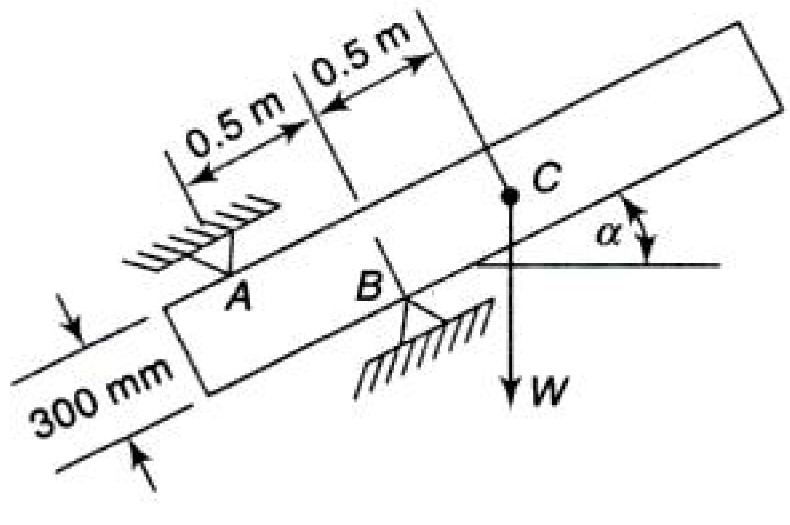
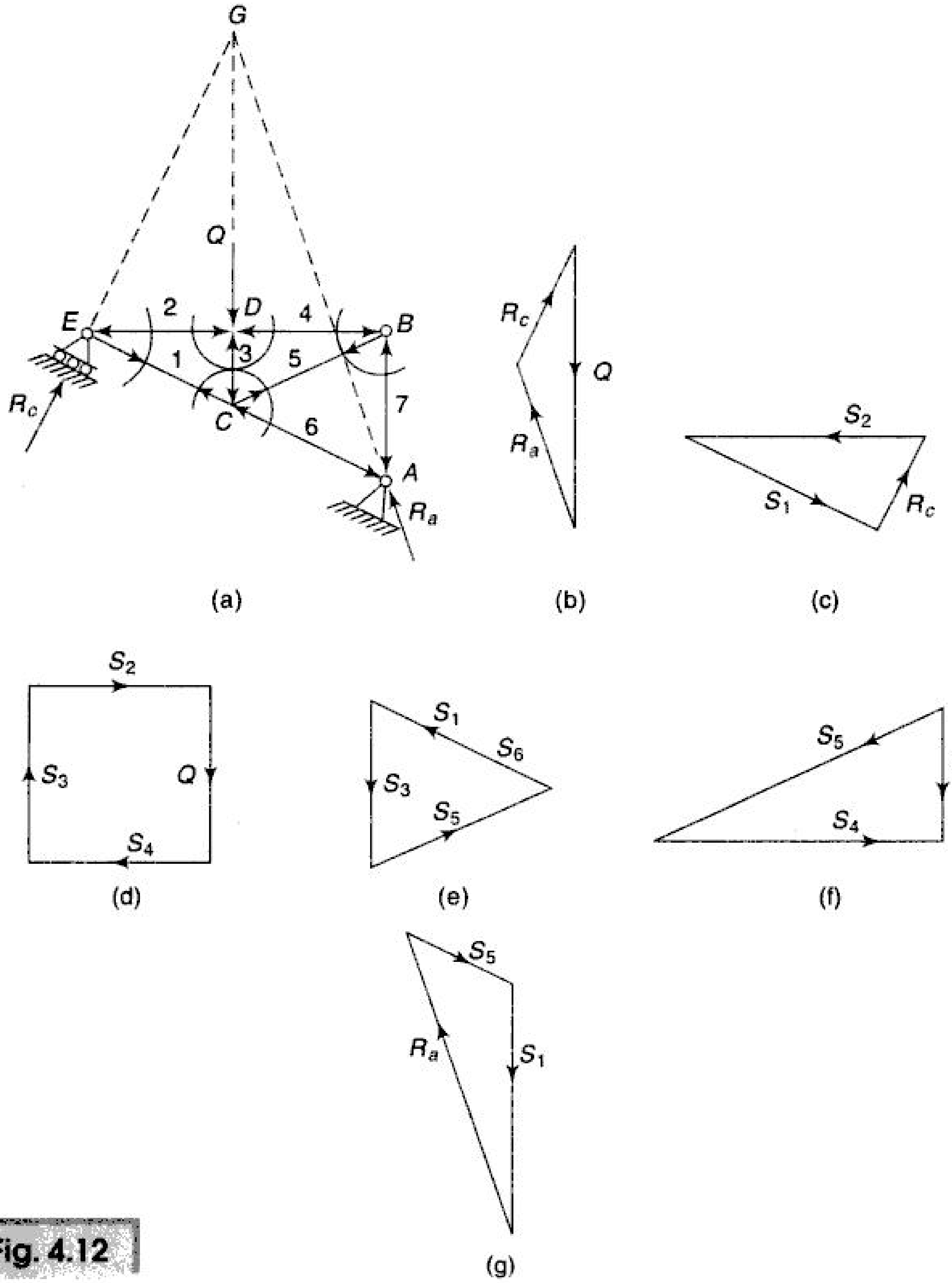


Fig. L



(a) (b) (c)

(d) (e) (f)

(g)

Fig. 4.12

Let us consider next the equilibrium of the hinge D. We first replace the bar 2 by its reaction S_2 equal but opposite to the previously considered reaction of this bar on the hinge E. There remain then but two unknown forces at D, representing the reactions on this hinge of the bars 3 and 4. Again we do not know whether these last two forces are directed toward the hinge or away from it, but knowing the directions of their lines of action we can construct the polygon of forces shown in Fig. 4.13, from which the forces, S_3 and S_4 are determined as before. In this case we see from the arrows on the sides of the polygon of forces that both bars are pressing on the hinge and hence are in compression. Arrows indicating such reactions can now be placed on the axes of the bars 3 and 4 at D, as shown in Fig. 4.12(a).

PRACTICE SET 4.3

Review Questions

1. Define plane truss.
2. State the assumptions made in the analysis of trusses.
3. Explain the method of joints for analysis of the truss.
4. Explain the importance of the assumptions in the analysis of truss.

Objective Questions

1. Minimum number of members required to form a simple truss
   (a) 2 (b) 3 (c) 4 (d) 5
   [*Ans.* (b)]
2. In the method of joints for the analysis of forces in the members of the truss, the number of equilibrium equations available at each joint are
   (a) 2 (b) 3 (c) 4 (d) 5
   [*Ans.* (a)]
3. Which of the following assumptions are necessary in the analysis of truss?
   I. The bars are connected at their ends by frictionless hinges.
   II. The bars are weightless.
   III. The axes of all bars lie in one plane, called the middle plane of the truss.
   IV. All forces acting on the truss are applied at the hinges only and also lie in the plane of the truss.
   The correct answer is
   (a) Only I (b) Only II
   (c) I, II and III (d) All of the above
   [*Ans.* (d)]

PROBLEM SET 4.3

1. Calculate the axial force S_i in each bar of the simple truss supported and loaded as shown in Fig. A. The triangle *ACB* is isosceles with 30° angles at *A* and *B* and $P = 5$ kN. (*Ans.* $S_1 = -3.34$ kN, $S_2 = -6.67$ kN; $S_3 = +5.77$ kN; $S_4 = +2.89$ kN; $S_5 = +5.77$ kN)

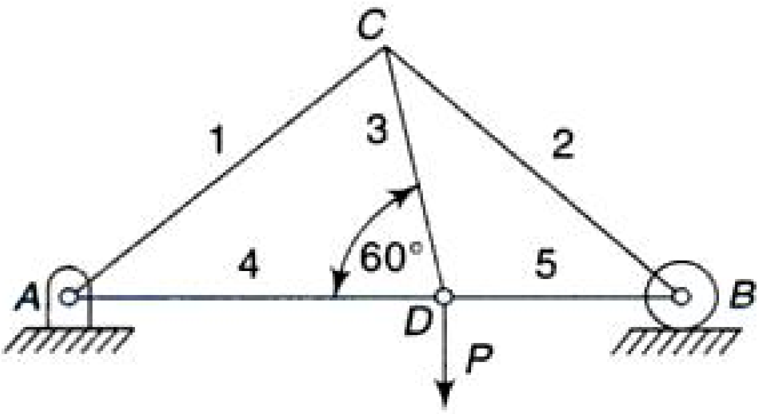


Fig. A

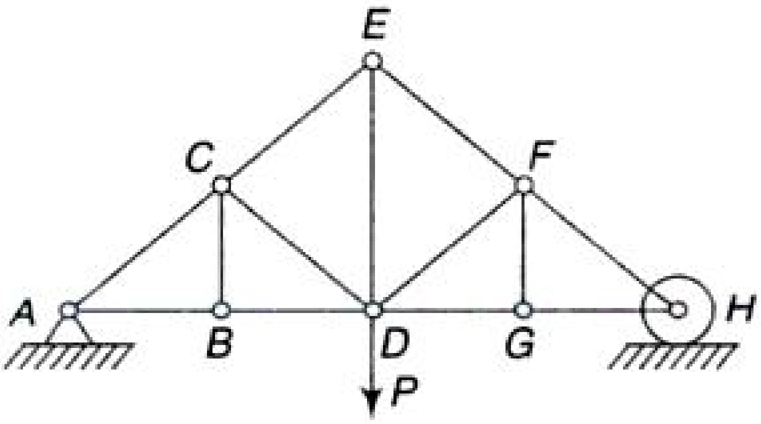


Fig. B

2. Prove that a tensile force equal to the applied load *P* is produced in the bar *DE* of the truss shown in Fig. B.
3. Determine the axial forces in the bars 1, 2, 3, 4 and 5 of the plane truss supported and loaded as shown in Fig. C.
   (*Ans.* $S_1 = -P$; $S_2 = +P$; $S_3 = -0.5P$; $S_4 = +0.442P$; $S_5 = -0.333P$)

The foregoing procedure in the analysis of trusses is called the *method of sections*. It consists essentially in the isolation of a portion of the truss by a section is such a way as to cause those internal forces that we wish to evaluate to become external forces on the isolated free body. By this procedure, we usually arrive at the case of equilibrium of general system of forces in a plane and the usual conditions of equilibrium [Eqs (18) or (19)] may be employed to evaluate the unknown forces as was done above. The success or failure of the method rests entirely upon the choice of section. In general, a section should cut only three bars, since only three unknowns can be determined from three equations of equilibrium. However, there are special cases where we may successfully cut more than three bars, and some of these will be illustrated in the following examples. Here we shall always assume tension to be positive and compression, negative.

Examples Examples Examples Examples Examples

1. Using the method of sections, determine the axial forces in bars for which 1, 2 and 3 of the tower loaded as shown in Fig. 4.15(a).

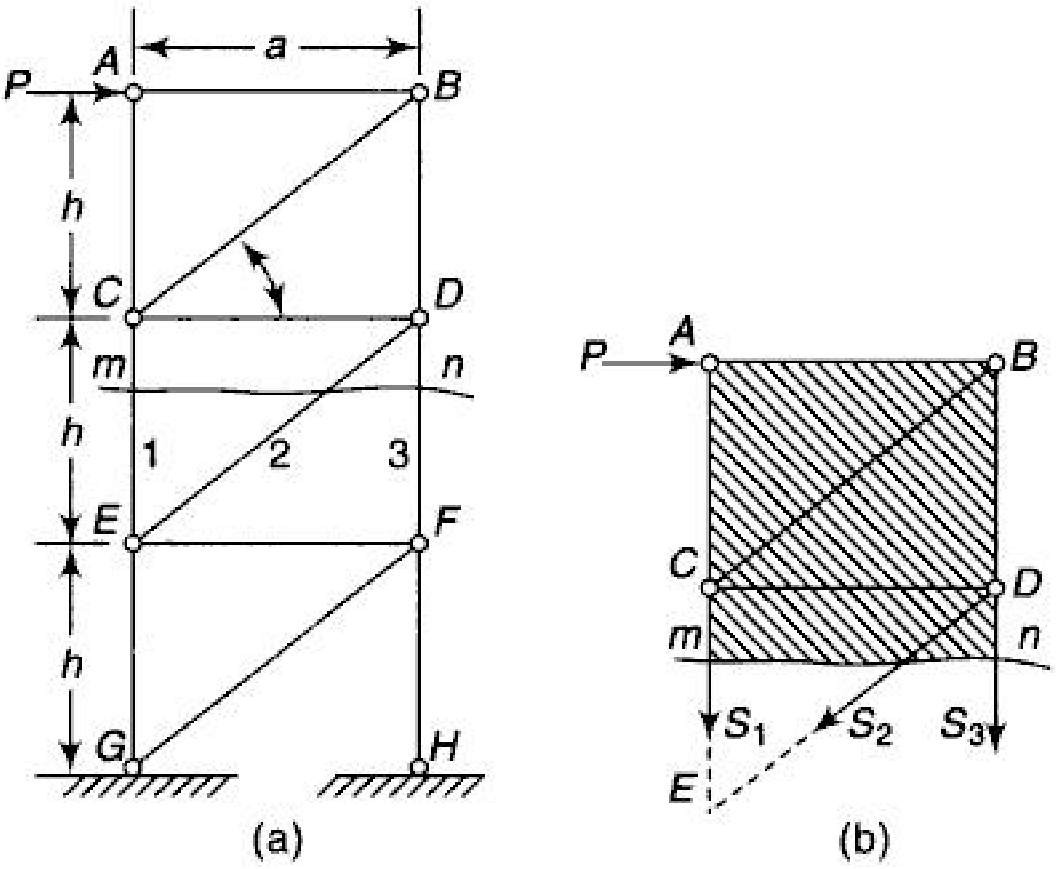


Fig. 4.15 (a) (b)

Solution: We begin by making a section *mn*, cutting the three bars for which the axial forces are required and consider the portion of the tower above this section as a free body [Fig. 4.15(b)]. Acting on this free body, we have the applied force P and the forces S_1, S_2, and S_3 representing the axial forces in the cut bars 1, 2 and 3.

Equating to zero the algebraic sum of moments of these forces with respect to point D, we obtain

$$S_1 a - 2Ph = 0$$

From which $S_1 = +Ph/a$, tension. Similarly, with E as a moment center, we have

$$-S_3 a - 2Ph = 0$$

6. Determine, by the method of sections, the axial force in each of the bars, 1, 2 and 3 of the plane truss shown in Fig. F.

(*Ans.* $S_1 = -1.5$ kN; $S_2 = -3.33$ kN; $S_3 = +2.665$ kN)

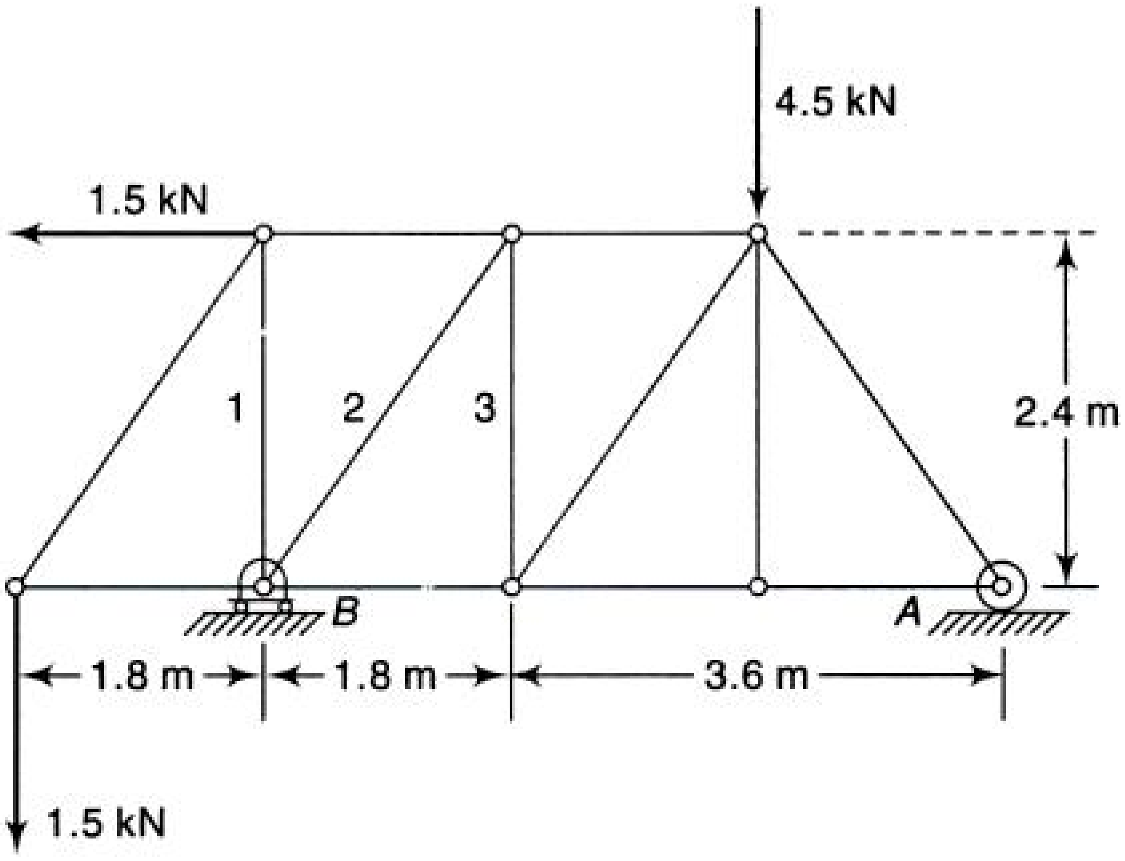

4.5 kN
1.5 kN
1
2
3
2.4 m
B
A
1.8 m
1.8 m
3.6 m
1.5 kN

Fig. F

7. Using the method of sections, calculate the axial force in each of the bars 1, 2 and 3 of the plane cantilever truss loaded as shown in Fig. G.

(*Ans.* $S_1 = -5.33P$; $S_2 = +2P$; $S_3 = -1.67P$)

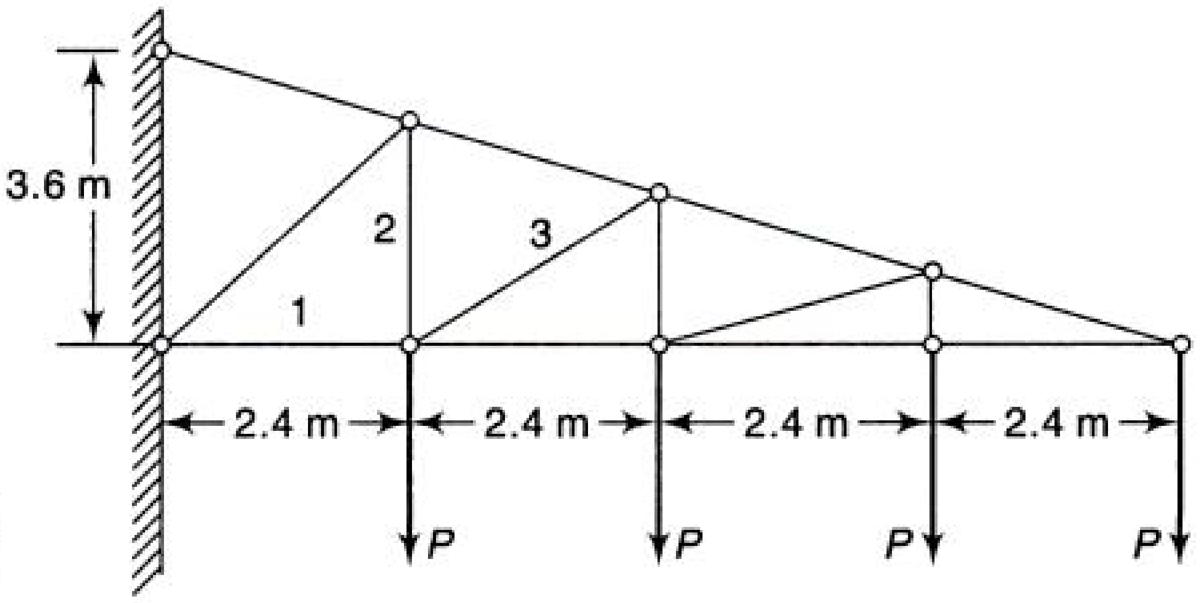

3.6 m
2
3
1
2.4 m
2.4 m
2.4 m
2.4 m
P
P
P
P

Fig. G

8. Determine the forces bars 1, 2 and 3 of the plane truss loaded and supported as shown in Fig. H. (*Ans.* $S_1 = -4Pa/h$; $S_2 = -P/2$; $S_3 = +4Pa/h$)

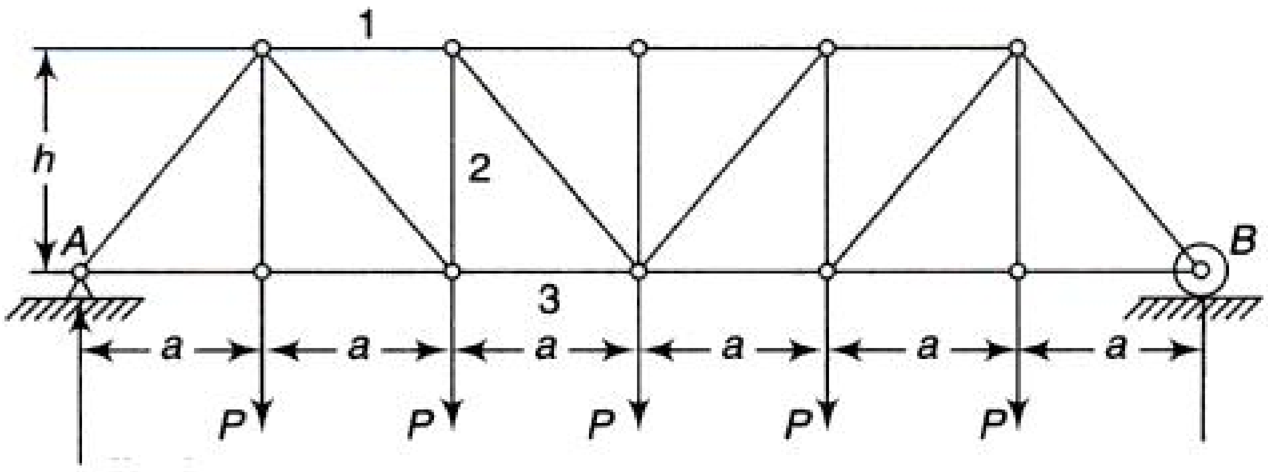

1
h
2
A
B
3
a
a
a
a
a
a
P
P
P
P
P

Fig. H

9. Determine the forces bars 1, 2 and 3 of the plane truss loaded and supported as shown in Fig. I. (*Ans.* $S_1 = S_2 = S_3 = -3Q\,a/2h$)

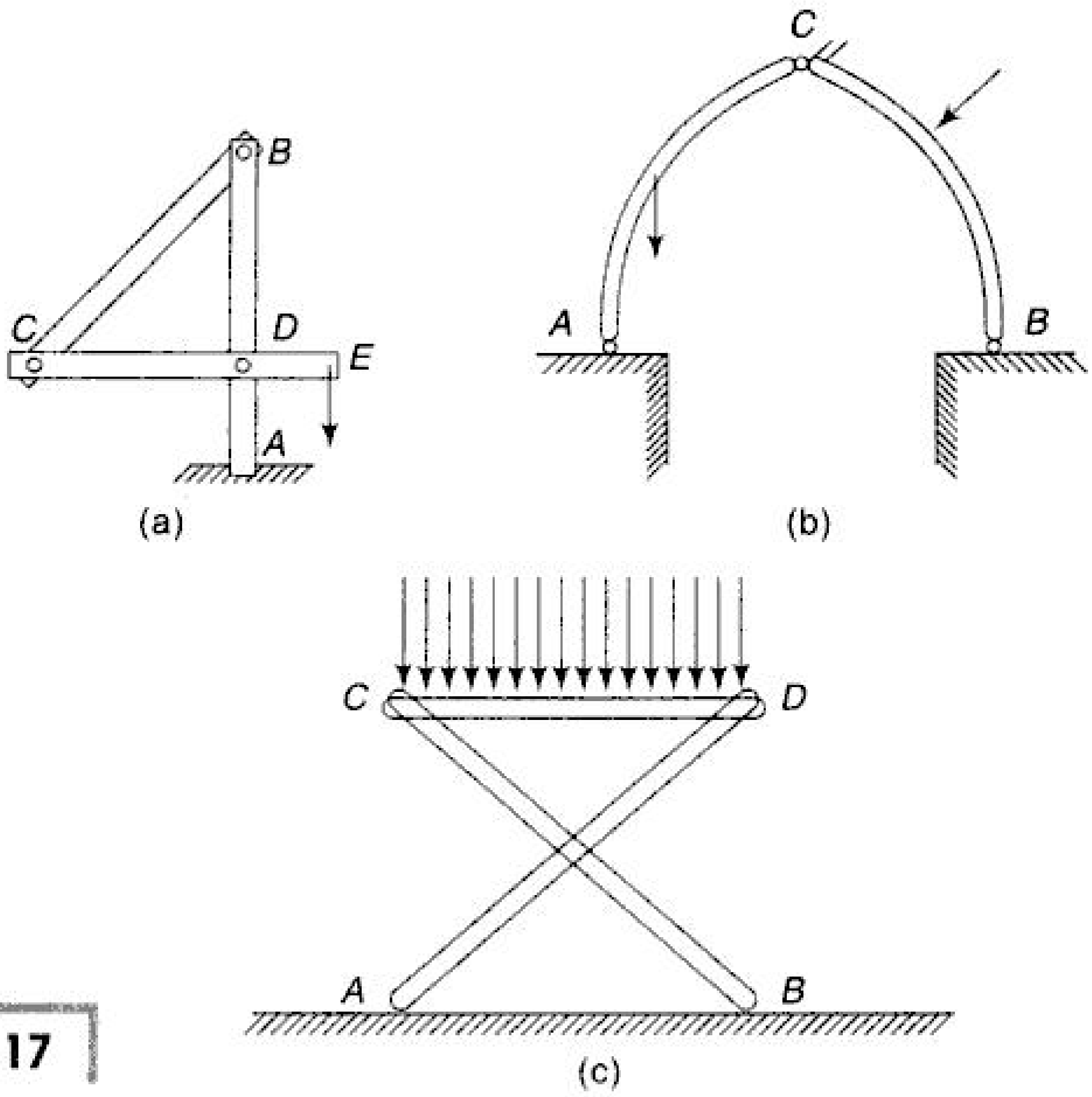


Fig. 4.17

This structure consists of two rigid bars or ribs AC and BC hinged together at C and to the foundation at A and B. Since each member is acted upon by a force at some intermediate point, it is subjected to bending, and the forces at its ends will not be directed along the axis of the member.

Accordingly, we do not know the direction of the reaction at either A or B; therefore we represent these reactions by rectangular components X_a, Y_a and X_b, Y_b as shown. Considering the equilibrium of the entire structure, we encounter a system of coplanar forces with four unknowns and with only three equations of equilibrium and the problem appears to be indeterminate. To avoid this difficulty, we disconnect the two bars at C and isolate each one as a free body, Fig. 4.18(b) and (c). Then on each of these free bodies we have, in addition to the forces already considered, a pair of rectangular components X_c, Y_c, representing the force transmitted from one bar to the other through the hinge C. If we assume these forces on the free body AC to be directed as shown in Fig. 4.18(b), then if follows from the law of action and reaction that the corresponding forces on BC must be oppositely directed, as shown in Fig. 4.18(c). It makes no difference which way we direct the forces X_c, Y_c, as long as they are opposite on the two free bodies. If either of them is incorrectly assumed, we shall simply obtain a result with negative sign after calculation.

We now have two simultaneous free-body diagrams [Fig. 4.18(b) and (c)] involving altogether six unknowns. Since for each free body we have three equations of equilibrium, Eqs (18) or (19), it follows that the problem is statically determinate, i.e., we have six equations and six unknowns. With points A and B as moment centers, Eq. (18) for the two systems will be as follows:

Solution: We begin with a free-body diagram of the entire frame, Eq. (18) may be written as follows:

$$-X_a + 2.5 = 0$$
$$Y_a - R_b - 2.5 = 0$$
$$-10\,R_b + 2.5 \times 5 + 2.5 \times 5 = 0$$

from which we obtain $X_a = 2.5$ kN, $Y_a = 7.5$ kN, $R_b = 5$ kN.

We now make a separate free-body diagram for each of the three members, as shown in Fig. 4.20(b) to (d). On these free-body diagrams, we place the given loads at *D* and *C* and the numerical values of the previously found reactions at *A* and *B*, as shown. At each of the point *E*, *F*, *G*, we indicate two rectangular components of force representing the interactions between members at these points of connection, but we do not immediately attach any arrows or symbols to these vectors.

Now let us consider the equilibrium of the vertical bar *AC* in Fig. 4.20(b). At this time none of the four forces acting at *E* and *G* is known, but, nevertheless, we can find X_a and X_g and by writing $\Sigma M_g = 0$, which give

$$2.5 \times 5 + 2.5 \times 10 - 5\,X_e = 0$$
$$2.5 \times 5 + 2.5 \times 10 - 5\,X_g = 0$$

From these two equations, we obtain $X_c = 7.5$ kN, $X_g = 7.5$ kN, directed as assumed. Accordingly, we at once record 7.5 kN to the left at *E* in Fig. 4.20(c) and 7.5 kN to the right at *G* in Fig. 4.20(d).

Next, we consider the equilibrium of the horizontal bar *DF* in Fig. 4.20(c). With *F* as a moment center, the equations of equilibrium Eq. (18) for this free body become

$$X_f - 7.5 = 0$$
$$Y_e - Y_f - 2.5 = 0$$
$$5\,Y_e - 2.5 \times 10 = 0$$

from which $X_f = 7.5$ kN, $Y_e = 5$ kN and $Y_f = 2.5$ kN, all directed as assumed. These numerical values may now be recorded at *E* in Fig. 4.20(b) and at *F* in Fig. 4.20(d), as shown.

Returning to the free body in Fig. 4.20(d), and writing $\Sigma Y_i = 0$, we obtain

$$7.5 - 5 - Y_g = 0$$

from which $Y_g = 2.5$ kN directed down. We immediately place a corresponding 2.5 kN force up at *G* in Fig. 4.20(d), and we see that all forces have now been evaluated.

As a check on our arithmetic, we observe that the forces on the inclined bar in Fig. 4.20(d), which have all been found from considerations of the other free-body diagrams, satisfy the three conditions of equilibrium for this bar.

Important Terms and Concepts

Plane frame Flexure members Method of members Three hinged frame

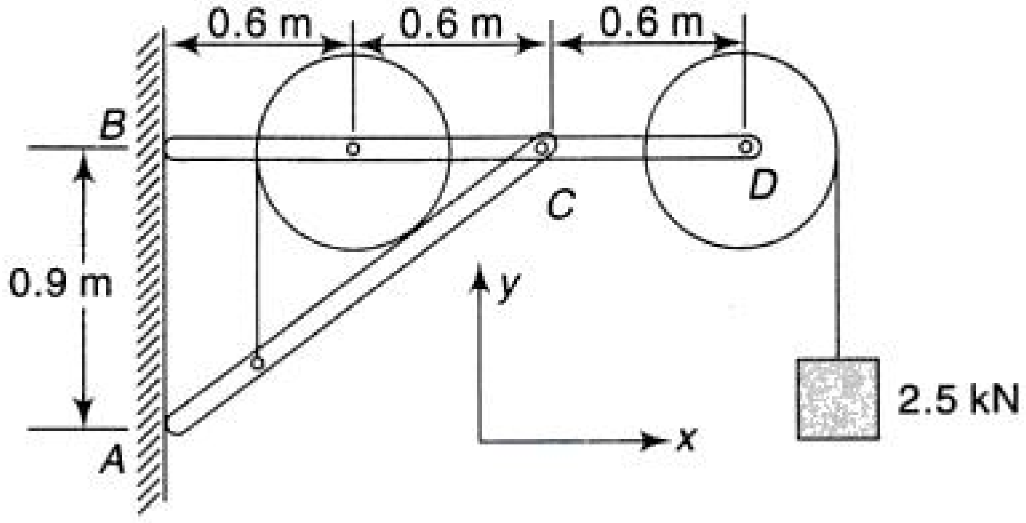

0.6 m
0.6 m
0.6 m
B
0.9 m
A
C
D
y
x
2.5 kN

Fig. H

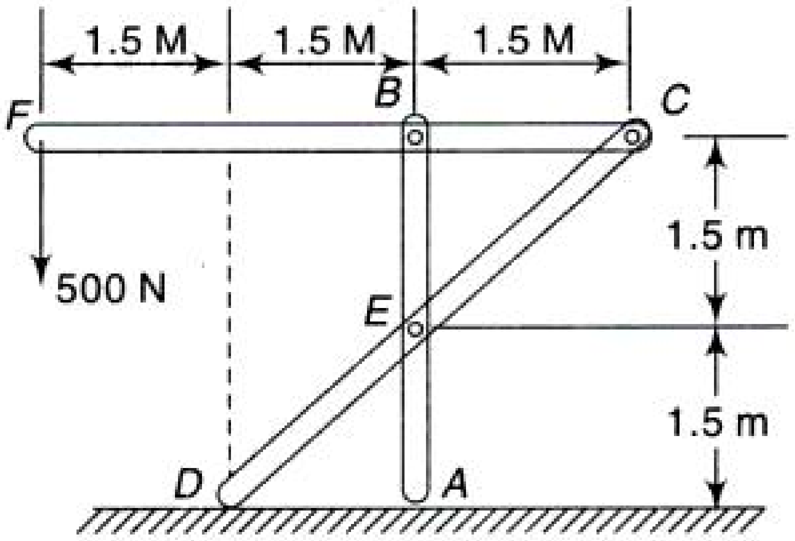

1.5 M
1.5 M
1.5 M
F
B
C
500 N
E
1.5 m
1.5 m
D
A

Fig. I

10. Calculate the pressure exerted against the sides of an ice cube of weight $W = 500$ N by the points A and B of the tongs which support it (Fig. J). The dimensions of the tongs are as follows: $a = 100$ mm, $b = 200$ mm, $c = 200$ mm, $d = 400$ mm. (*Ans.* $R_a = R_b = 475$ N)
11. Calculate the shear force R_c on the pin C of the ice tongs in Fig. J. (*Ans.* $R_c = 837$ N)

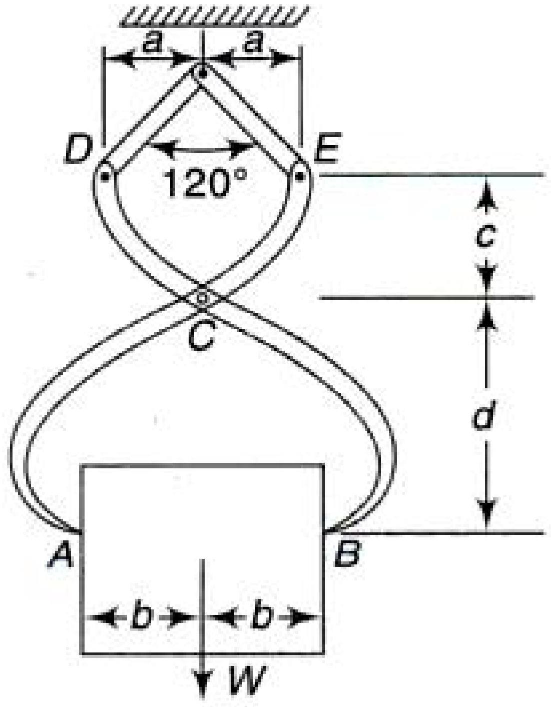

a
a
D
E
120°
c
C
d
A
B
b
b
W

Fig. J

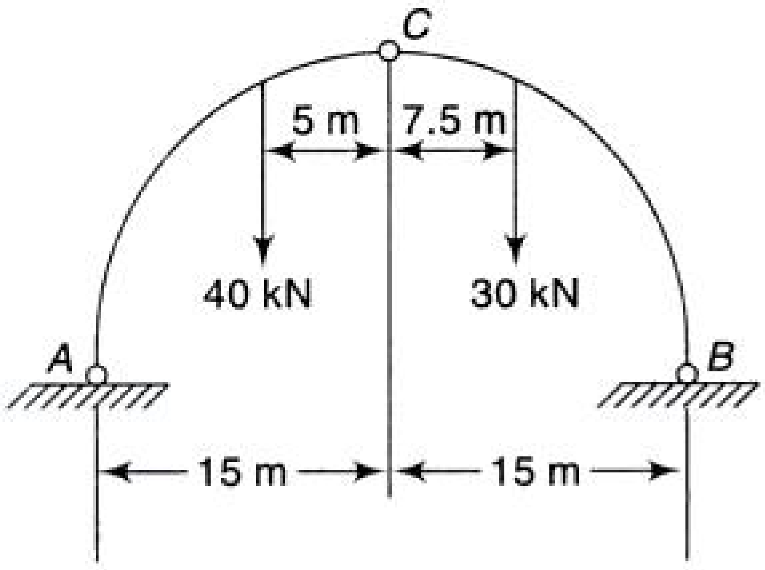

C
5 m
7.5 m
40 kN
30 kN
A
B
15 m
15 m

Fig. K

12. Find the reactions at the supports A and B of the semicircular three-hinged arch loaded as shown in Fig. K. (*Ans.* $R_a = 40.1$ kN, $R_b = 41.5$ kN)
13. Calculate the horizontal and vertical components of the reactions at A and B of the frame structure loaded as shown in Fig. L. (*Ans.* $X_a = 315$ N; $Y_a = 210$ N; $X_b = 885$ N; $Y_b = 390$ N)

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The same procedure may be used in the case of two unequal parallel forces acting in opposite directions, as shown in Fig. 4.23(a). In this case again, we begin with the polygon of forces *ABC*, select a pole *O*, and draw rays 1, 2 and 3. Starting with any point *a* in the plane of action of the given forces, the funicular polygon *abcd* with its sides parallel, respectively, to rays *OA*, *OB*, and *OC* is obtained as shown [Fig. 4.23(a)]. The magnitude and direction of the resultant are given by the closing side $\overline{AC}$ of the polygon of forces [Fig. 4.23(b)], and a point on its line of action, by the intersection *e* of the first and last sides of the funicular polygon [Fig. 4.23(a)].

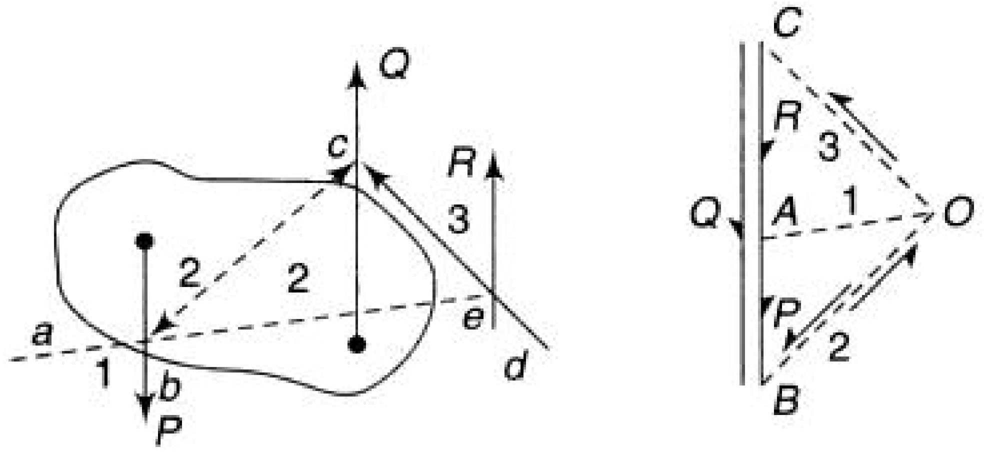

Fig. 4.23 (a) (b)

Let us consider now a general case of several coplanar forces F_1, … F_5 applied to a body as shown in Fig. 4.24(a). We begin with the construction of the polygon of forces *ABCDEF*. Choosing an arbitrary pole *O*, drawing the rays 1, 2, 3, 4, 5, 6 and constructing, in the plane of action of the forces, the lines *ab*, *bc*, … *fg*, parallel to these rays, we obtain the funicular polygon *abcdefg* as shown. At the apexes of this polygon, each of the given forces F_1, … F_5 is replaced by its two components as represented to rays in Fig. 4.24(b). The forces acting along the sides *bc*, *cd*, *de*, and *ef* are pairs of equal and opposite forces and may be removed from the system. There remain only the forces 1 and 6 acting at points *b* and *f* which are equivalent to the given forces F_1, … F_5. The magnitude and direction of the resultant of these forces are given by the closing side $\overline{AF}$ of the polygon of forces [Fig. 4.24(b)], and a point on its line of action, by the intersection *h* of the first and last sides of the funicular polygon [Fig. 4.24(a)].

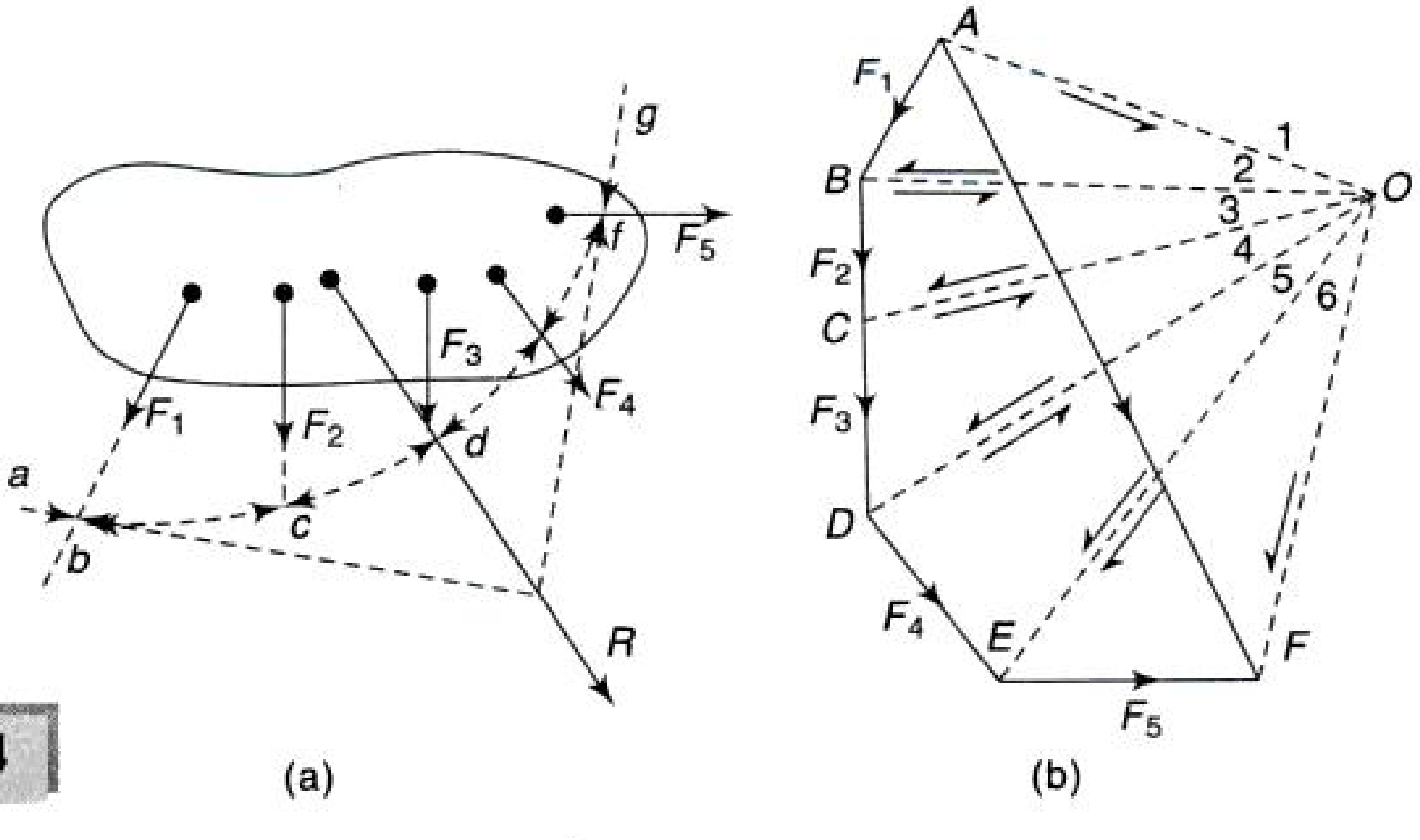

Fig. 4.24 (a) (b)

R_c and R_a, respectively, we obtain the apexes a and c of the funicular polygon and can draw its closing side ac after which the missing ray 3 in the polygon of forces can be drawn through point O parallel to this closing side ac of the funicular polygon. The intersection of ray 3 with the vertical line EF' determines the apex F of the closed polygon of forces. The vectors $\overline{EF}$ and $\overline{FD}$ represent, respectively, the reactions R_a and R_c.

It should be noted that, while any position of the point F along the line EF' [Fig. 4.27(b)] can satisfy the condition of a closed polygon of forces, only the point F as obtained above can simultaneously satisfy the condition of a closed funicular polygon.

3. Determine graphically the reactions R_a and R_b at the supports A and B of the truss loaded and supported as shown in Fig. 4.28(a).

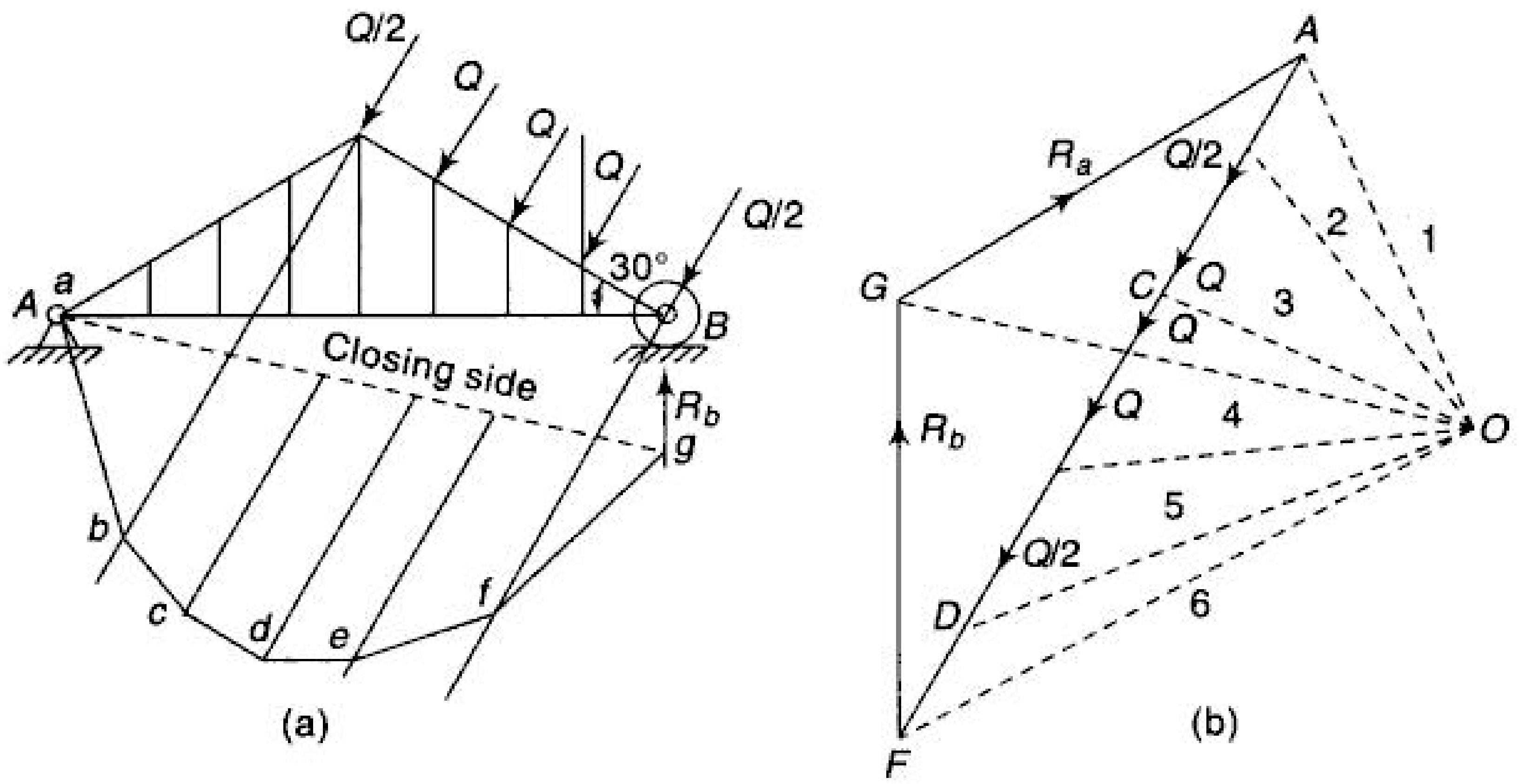

Fig. 4.28

Solution: Since the truss is in equilibrium under the action of the applied forces Q together with the reactions R_a and R_b, all these forces considered together must build a closed polygon of forces and their funicular polygon must be closed also. Proceeding on this basis, we begin with the construction of the polygon of forces [Fig. 4.28(b)]. The vectors representing the active forces Q are laid out in order, and from point F the vertical line FG' is extended in the known direction of the reaction R_b. Since, however, we know neither the magnitude of this reaction nor the direction of the reaction R_a, the closed polygon of forces cannot yet be completed. We know only that from some point G on the vertical line FG' the vector $\overline{GA}$ representing the reaction R_a must close the polygon. An arbitrary pole O is now selected and the rays 1, 2, 3, 4, 5 and 6 drawn, as shown in Fig. 4.28(b). The ray 7 cannot yet be drawn, since the apex G of the polygon of forces is not known at this stage of the construction.

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2. Find the resultant of the coplanar forces acting on the gravity dam section shown in Fig. B by constructing a funicular polygon. Forces are shown in kN.

(1 kN = 1000 N)

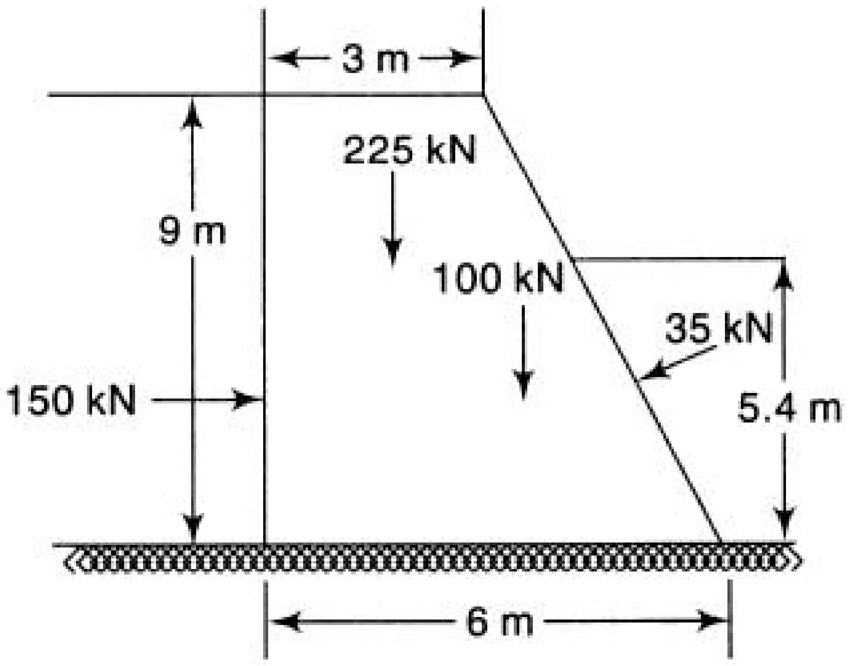
3 m
225 kN
9 m
100 kN
35 kN
150 kN
5.4 m
6 m

Fig. B

3. Determine, graphically, the reactions at the supports of a girder *AB* due to the locomotive loading shown in Fig. C. (*Ans.* R_a = 1042 kN, R_b = 1108 kN)

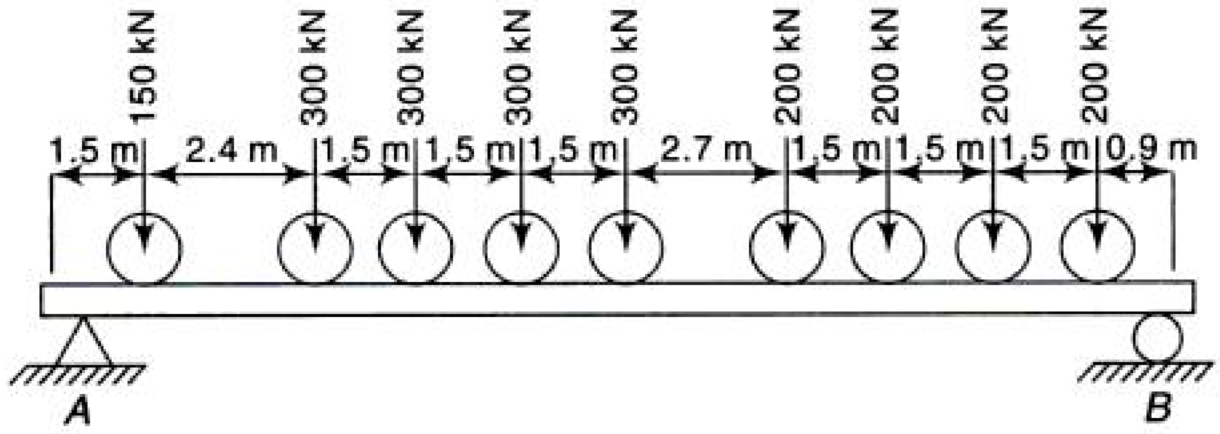
150 kN
300 kN
300 kN
300 kN
300 kN
200 kN
200 kN
200 kN
200 kN
1.5 m
2.4 m
1.5 m
1.5 m
1.5 m
2.7 m
1.5 m
1.5 m
1.5 m
0.9 m
A
B

Fig. C

4. Treating the flexible cable overrunning the pulleys in Fig. D as a funicular polygon, find graphically the forces exerted on the members of the frame by the axles of the pulleys *B* and *D*. Each stationary pulley is 300 mm in diameter.

(*Ans.* P_b = 23.4 kN ; P_d = 10.3 kN)

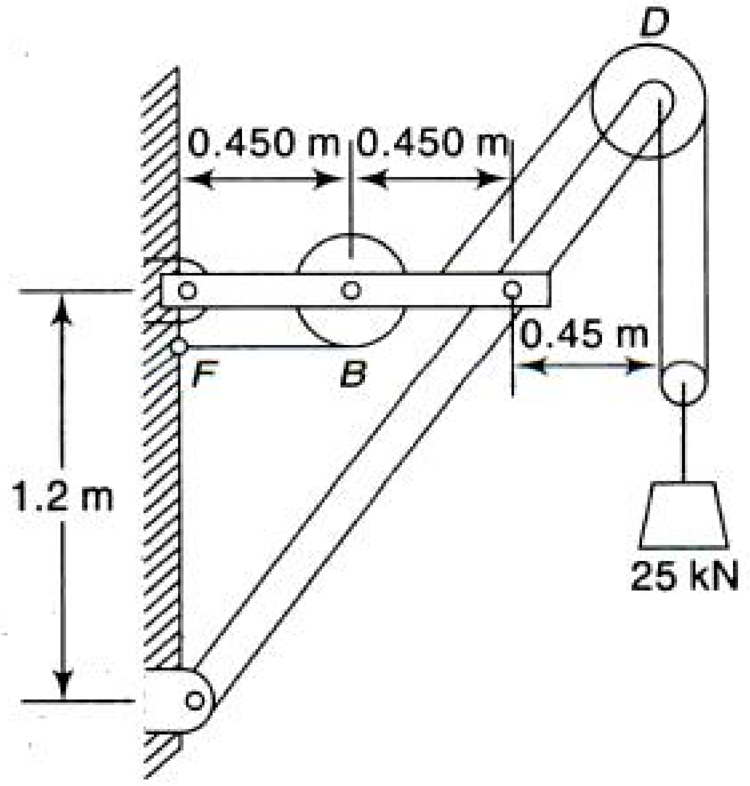
D
0.450 m
0.450 m
0.45 m
F
B
1.2 m
25 kN

Fig. D

5. Find graphically the reactions at *A* and *B* for the beam loaded as shown in Fig. E.

(*Ans.* R_a = 430 N ; R_b = 385 N)

6. Determine graphically the reactions R_a and R_b at the supports *A* and *B* of the horizontal beam *AB* due to the action of the vertical loads applied as shown in Fig. F.

(*Ans.* R_a = 2405 N ; R_b = 1745 N)

4.7 MAXWELL DIAGRAMS

We shall discuss here a graphical method of analysis of simple trusses based on the method of joints previously explained in Section 4.3. Referring to Fig. 4.30(a), we have a simple truss *ABC* supported at *A* and *B* and carrying a vertical load *P* at *C*. Thus the vertical reactions at *A* and *B* are each *P*/2 and we have the entire truss in equilibrium under the action of three parallel forces as shown. Denoting the axial forces in the bars by S_1, S_2, S_3 and considering the equilibrium of each of the joints *A*, *B*, *C*, in succession, we obtain the closed triangles of forces shown in Fig. 4.30(b) from which the magnitudes of the axial forces S_1, S_2, S_3 can be scaled.

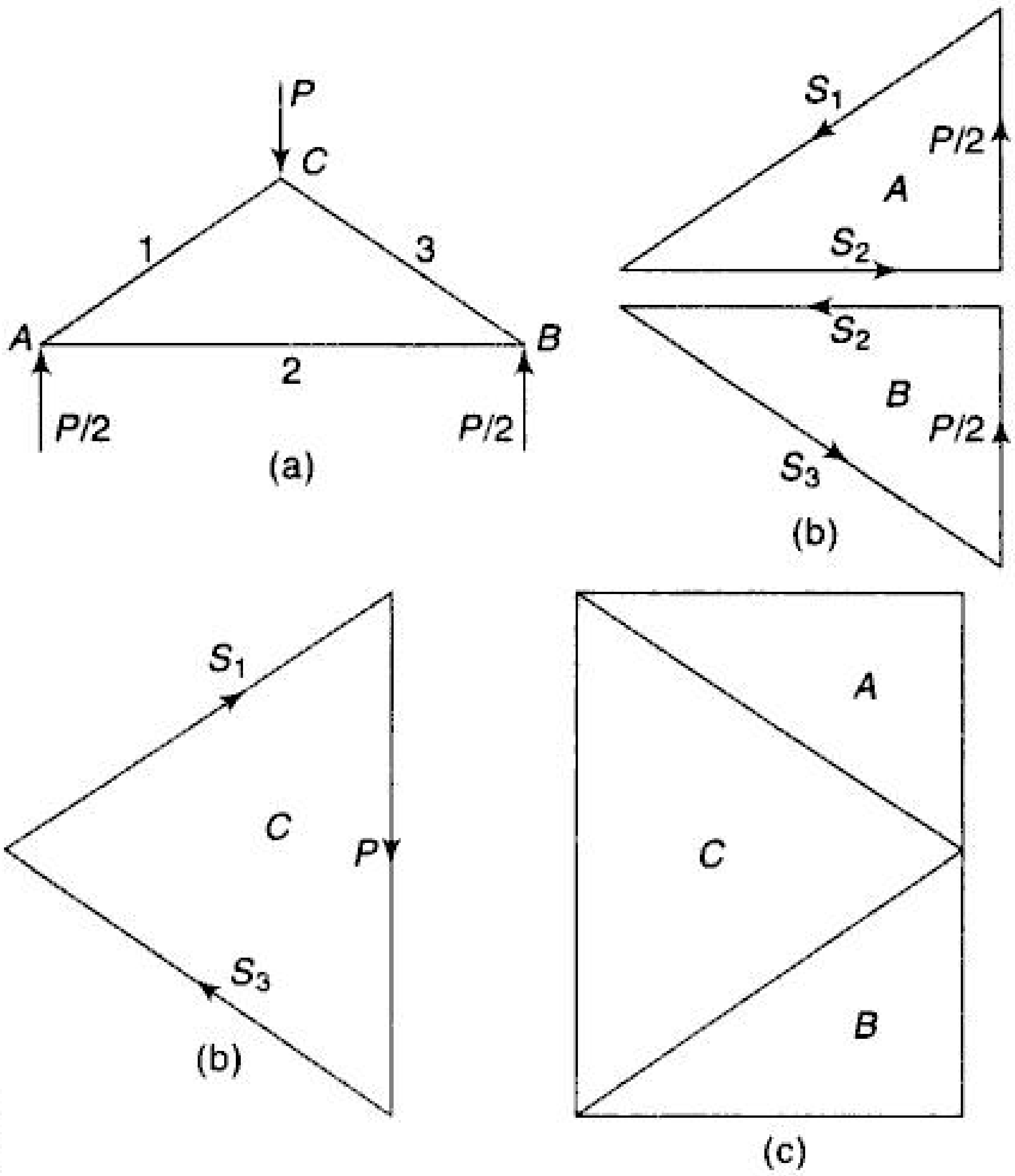

Fig. 4.30

We note now that each of these axial force vectors appears in two different polygons, once for each of the joints at the ends of that member. To avoid this duplication of vectors, the separate polygons of forces, under certain conditions, can be superimposed to form one composite diagram called a Maxwell diagram[1] for the truss. For example, the polygons of forces in Fig. 4.30(b), when superimposed, make the composite diagram shown in Fig. 4.30(c). Such superposition is desirable, since it reduces the amount of necessary construction and makes a more compact record of the analysis.

In constructing separate polygons of forces for the various hinges of a truss, it makes no difference in what order we add the vectors. Each hinge is in equilibrium, and the forces acting on it must build a closed polygon in any order. However, if we wish to superimposed these polygons as in Fig. 4.30(c), they must be constructed in a definite manner. It may be noted now that, for each polygon in

explained, and we obtain finally the composite diagram, as shown in Fig. 4.32(b). This diagram is drawn to the scale km = 40 kN and from it the axial force in any bar of the truss may be found. The student will find it a worthwhile exercise to follow through the diagram and ascertain for each bar whether it is in tension or compression.

2. A crane supports a load P which hangs on a flexible cable overrunning small pulleys attached to the top chord joints, as shown in Fig. 4.33(a). Find graphically the axial forces produced in the various bars.

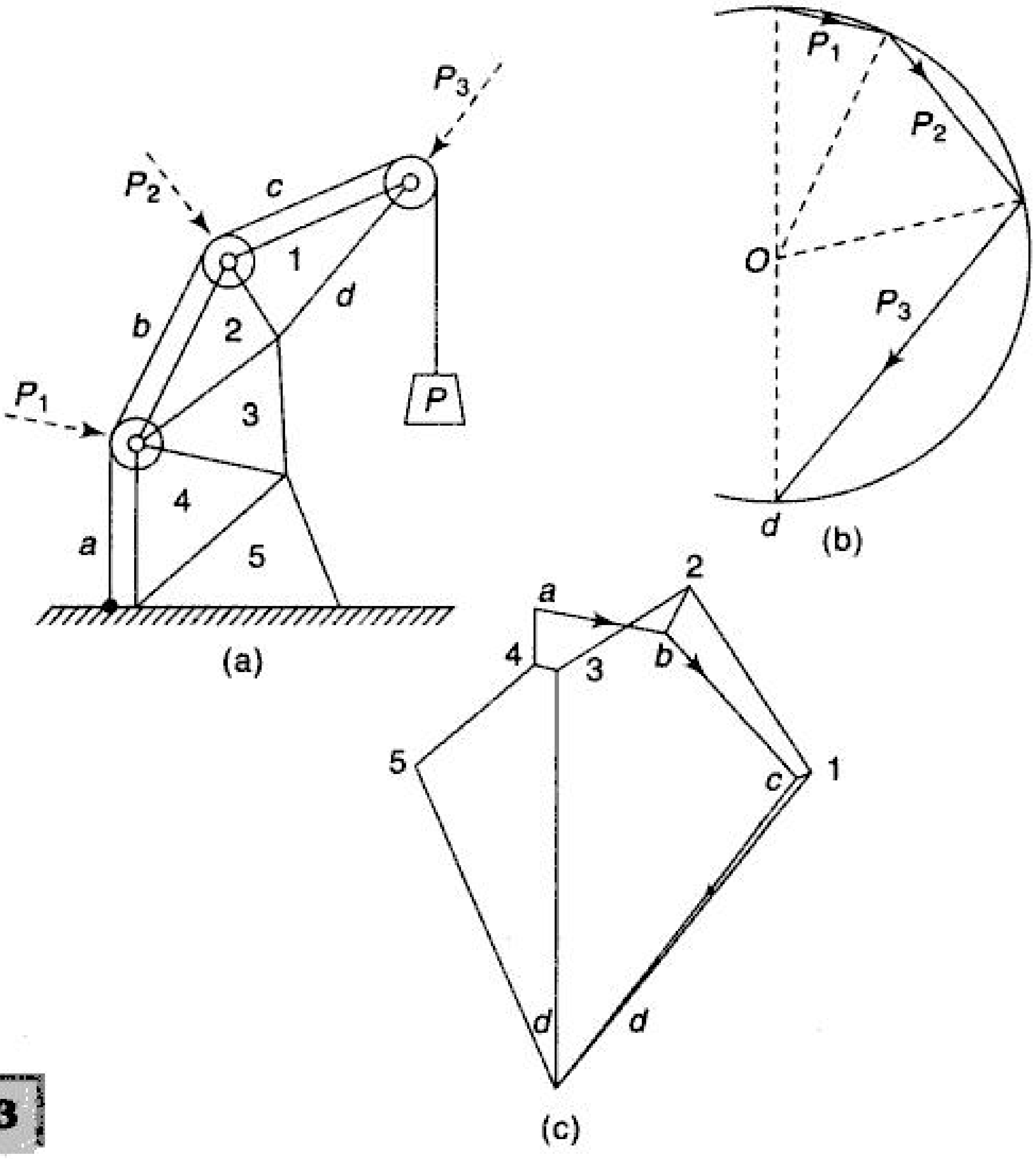

Fig. 4.33

Solution: We assume that the diagram of the truss in Fig. 4.33(a) has been drawn to scale. Then to determine the internal forces in the bars, we need to know first the magnitudes and directions of the external forces P_1, P_2, P_3 exerted on the joints by the axles of the pulleys. These, we find by observing that the cable, under uniform tension P, represents a funicular polygon for the desired forces, as already discussed in Example 1 of Section 4.6. Accordingly, in Fig. 4.33(b), we construct a circle the radius of which represents, to some convenient scale, the magnitude of the load P. Then drawing rays Oa, Ob, Oc and Od, parallel to the several portions of the cable, we find the forces P_1, P_2, P_3 exerted on the joints of the truss, as shown. We place these forces on the truss (thereafter ignoring the cable) and letter (or

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which, by using Eqs (e) and (f), becomes,

$$dS = \mu S \, d\theta$$

or

$$\frac{dS}{S} = \mu \, d\theta \tag{g}$$

This expresses the ratio between the increment of tension over the length of the element to the total tension in the belt at the point defined by the angle θ. Integrating Eq. (g) over the entire line of contact *AB*, of length $r\beta$, we obtain

$$\ln \frac{S_1}{S_2} = \mu\beta \text{ or } \qquad \frac{S_1}{S_2} = e^{\mu\beta} \tag{h}$$

We see that the ratio between the tensions $S_1$and S_2 in the belt on the two sides of the pulley increases very rapidly with the magnitude of the central angle β of the line of contact *AB*. This explains how a man hold a great load on the end of a rope simply by taking a turn of two of rope around a post. It will also be noted that the ratio S_1/S_2 is independent of the radius *r* of the pulley.

Center of Pressure

Another problem involving distributed force is illustrated in Fig. 4.38. The opening to a penstock through the base of a gravity dam is closed by a circular plate *AB* and it is desired to define the resultant water pressure *P* on the plate. This is really a problem involving force distributed over an area, but we can easily bring it to an equivalent distribution in the vertical plane of the diameter *AB* of the plate.

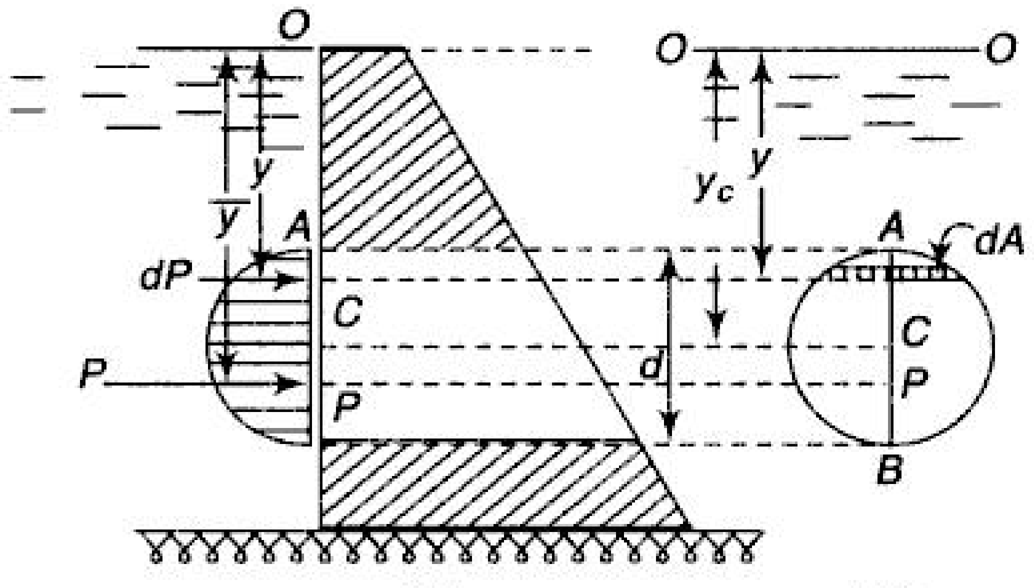


Fig. 4.38

Looking directly at the plane of the plate [Fig. 4.38(b)], we consider one element of area *dA* in the form of a horizontal strip at depth *y* as shown. Then since the intensity of water pressure $p = wy$ is constant across the length of this strip, the corresponding element of force $dP = wy\,dA$ acts at the mid-point of the strip, i.e. on the vertical diameter *AB*. This conclusion holds for each such horizontal strip, and we obtain a series of elemental forces distributed along *AB* and all lying in the same vertical plane, Fig. 4.38(a). The point *P* through which the resultant pressure acts on the plate is called the *center of pressure* and we need only to find the depth $\bar{y}$ to completely define its position on the vertical diameter *AB*.

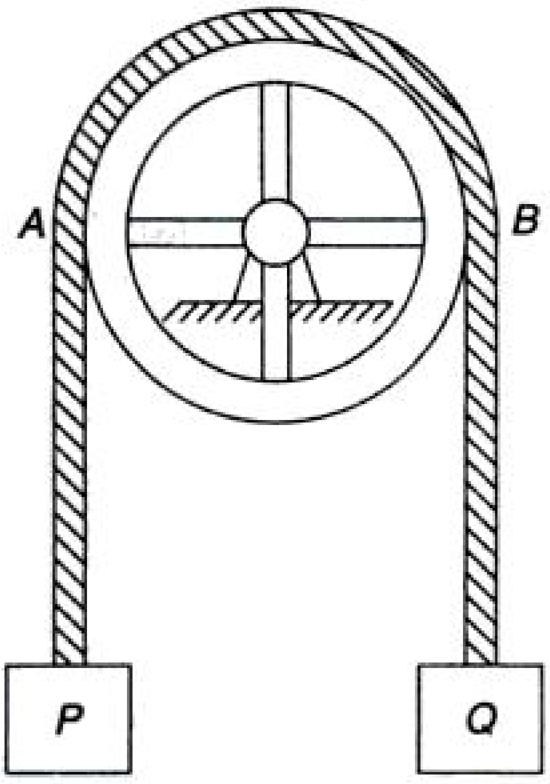


Fig. C

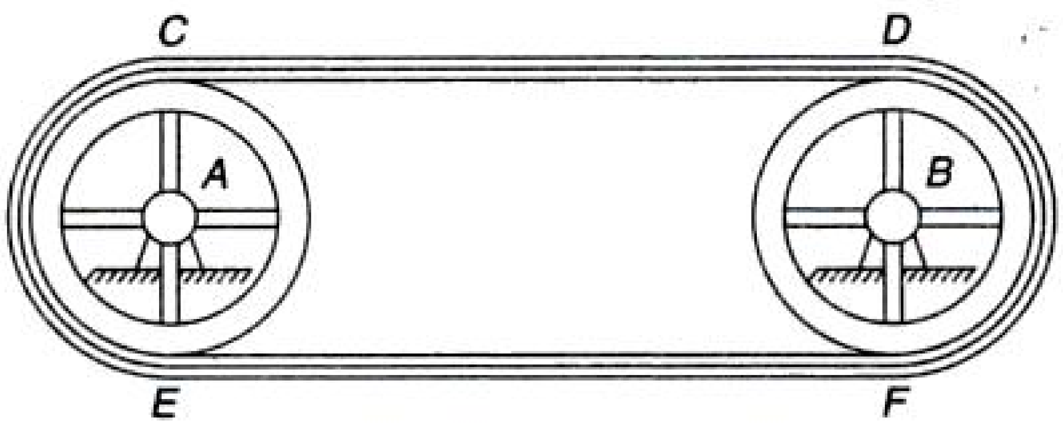


Fig. D

4.9 FLEXIBLE SUSPENSION CABLES

In engineering structures, we sometimes encounter flexible cables or chains suspended between two supports at their ends and subjected to the action of vertical load continuously distributed along their lengths. The distribution of load may, in general, be uniform or otherwise. Two of the most commonly encoutered cases of loading are illustrated in Fig. 4.39. Figure 4.39(a) represents the case of a flexible chain freely suspended in the gravity field and subjected to the action of its own distributed weight only. Such loading is, of course, uniformly distributed with respect to the curve of the chain itself. Figure 4.39(b) represents the case of a thin wire cable or cord subjected to the action of a uniformly distributed load attached to it by vertical *hangers*. In the event that this loading is large compared with the weight of the cable itself, we may assume in this case that the load is uniformly distributed with respect to the horizontal span.

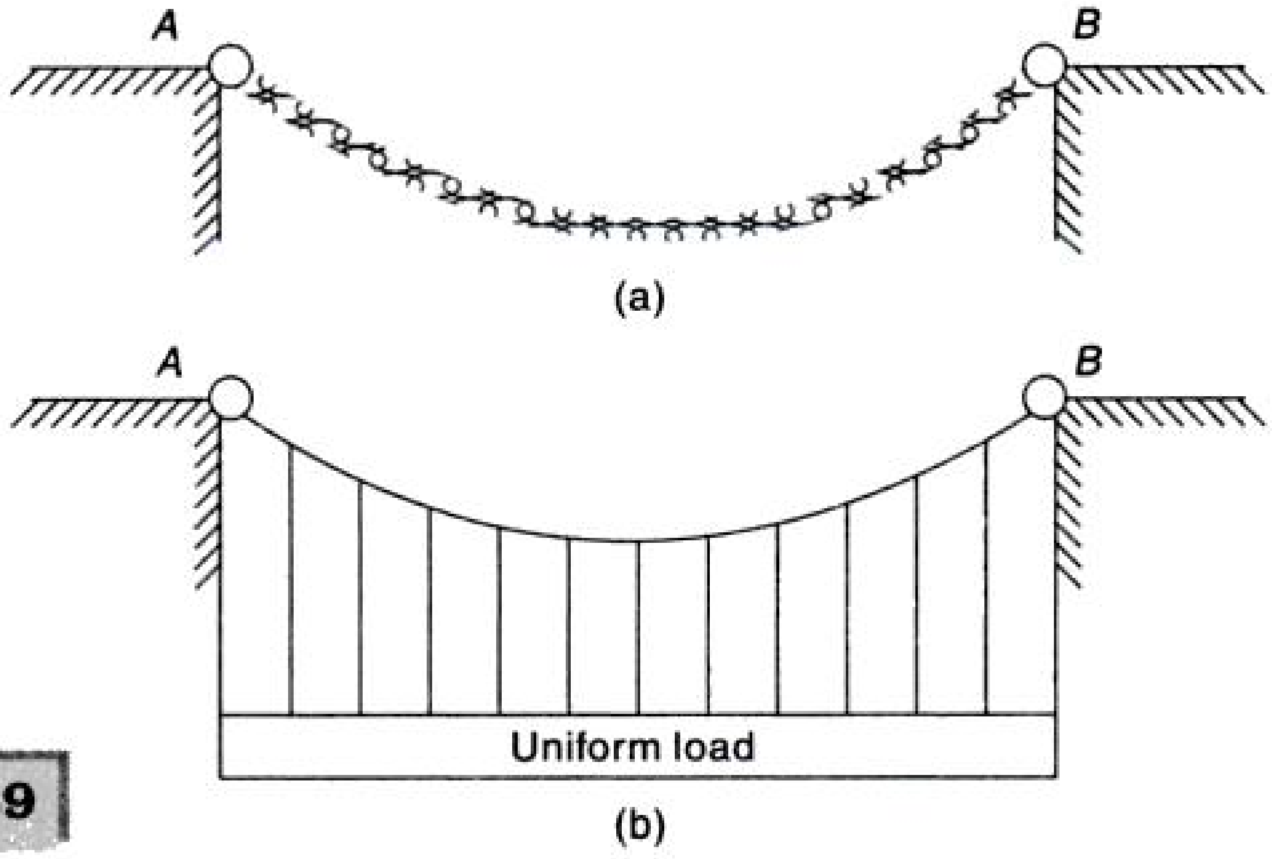


Fig. 4.39

tension H in the cable may be found at once, after which any of the quantities previously defined in terms of H can be calculated without difficulty.

Catenary Cable

Assuming that the cable in Fig. 4.42 hangs freely in the gravity field and is subjected only to its own weight uniformly distributed along the curve, Eq. (a) becomes

$$\frac{dy}{dx} = \frac{qs}{H} \tag{k}$$

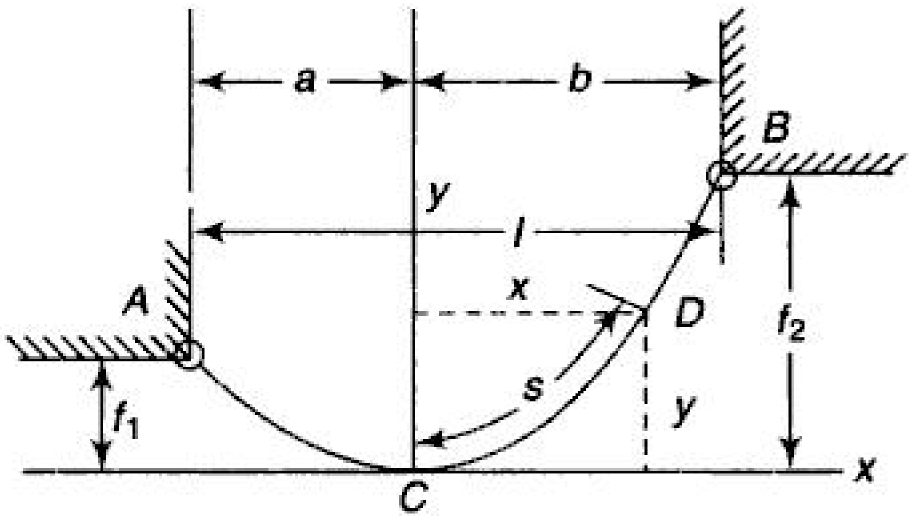


Fig. 4.42

where q is the weight per unit length of the cable and s is the length of the arc CD. Before this equation can be integrated, it will be necessary to express the length S as a function of the coordinates x and y. To do this, we use the relationship,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

which, with the above value of dy/dx, becomes

$$ds = \sqrt{1 + \left(\frac{qx}{H}\right)^2}\, dx$$

Integration of this last equation gives

$$\frac{H}{q} \text{arc sinh } \frac{qs}{H} = x + C_1$$

where C_1 is a constant.
Since, for the coordinate axes as shown in Fig. 4.42, we have $s = 0$ when $x = 0$, it is evident that $C_1 = 0$ and the above equation becomes

$$s = \frac{H}{q} \sinh \frac{qx}{H} \tag{l}$$

Substituting the value of s from Eq. (1) into Eq. (k), we obtain

$$dy = \sinh \frac{qx}{H}\, dx \tag{m}$$

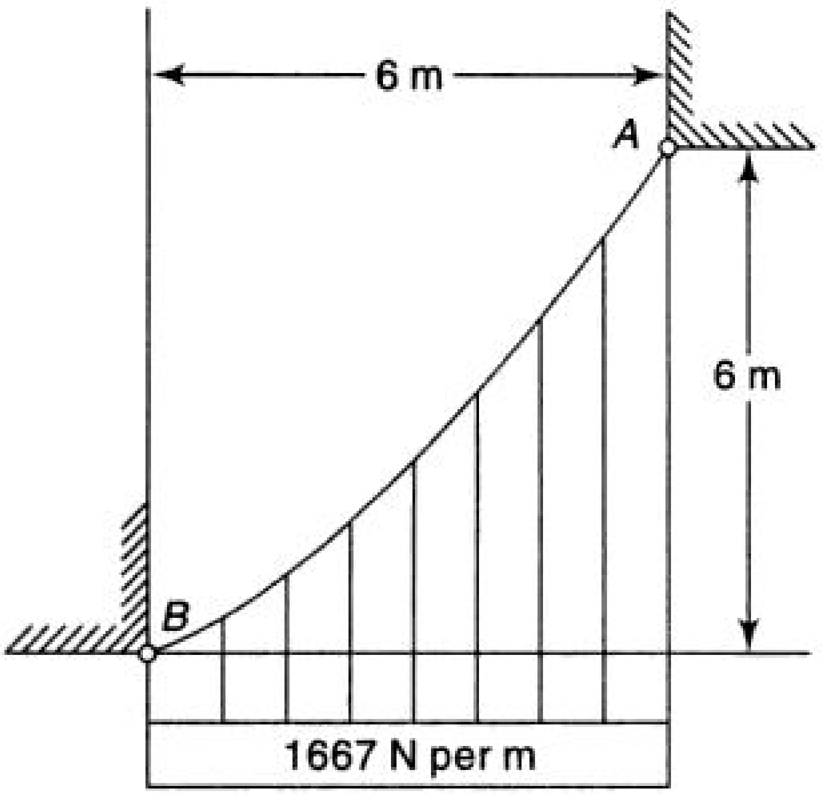


Fig. D

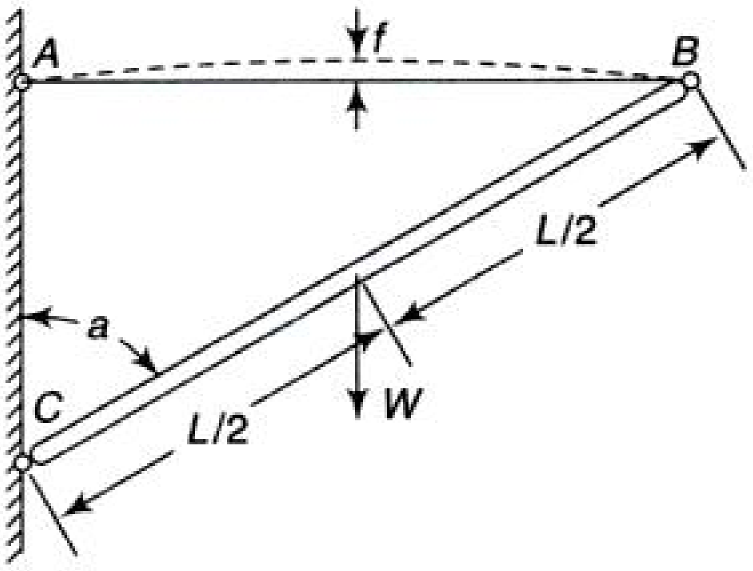
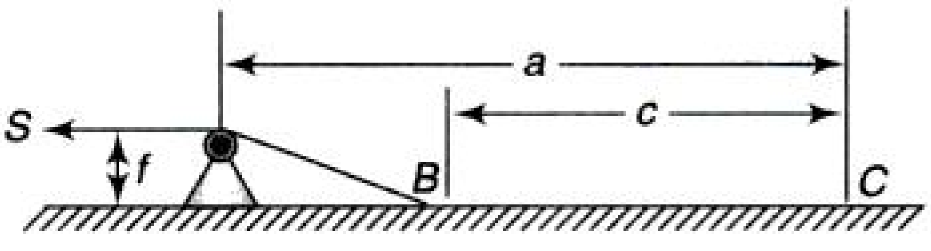


Fig. E

8. A flexible cable 30 m long and weighing 38 N per metre of length is freely suspended at its ends from two supports 15 m apart and having the same elevation. Find the sag f at the middle of the span. (*Ans.* $f = 11.7$ mm)
9. A flexible cable of uniform weight per unit length rests partly on a horizontal plane and passes over a small pulley at A, as shown in Fig. F. By gradually increasing the force S applied to the end of the cable, the length of contact BC with the plane diminishes to a certain limiting value c at which sliding of the cable along the plane impends. Find this limiting value c if $a = 60$ m, $f = 6$ m and the coefficient of friction between the cable and the plane is $\mu = 0.5$.

(*Ans.* $c = 44.1$ m)

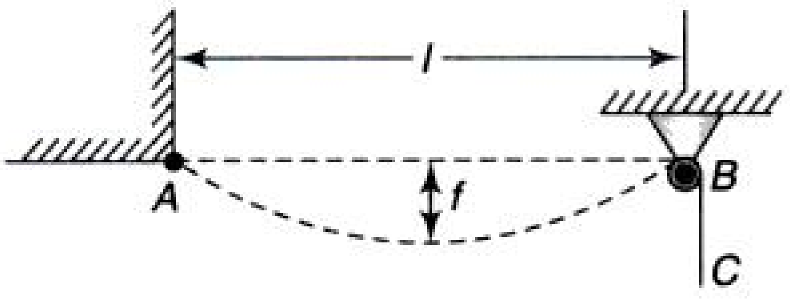
Fig. F



10. Determine the minimum length L of a flexible cable AC of uniform weight per unit length which can hang in equilibrium as shown in Fig. G. Neglect friction and the dimensions of the pulley B. What is the sag-span ratio?

(*Ans.* $L_{m/n} = 1.14\,l + 0.80\,l = 1.94l; f\,/\,l = 0.238$)

Fig. G



lines of action of all but one of any number of concurrent forces in space lie in one plane, then equilibrium can exist only if this one force is zero. (4) If the known lines of action of all but two of any number of concurrent forces in space lie in one plane and one of these two forces is known in magnitude, then the magnitude of the other can always be found without difficulty. The proof of each of these statements is left to the student.

Examples Examples Examples Examples Examples

1. Three concurrent forces F_1, F_2 and F_3 have the lines of action OA, OB and OC as shown in Fig. 5.4, and the magnitudes shown in the following table. Find the magnitude and direction of their resultant R.

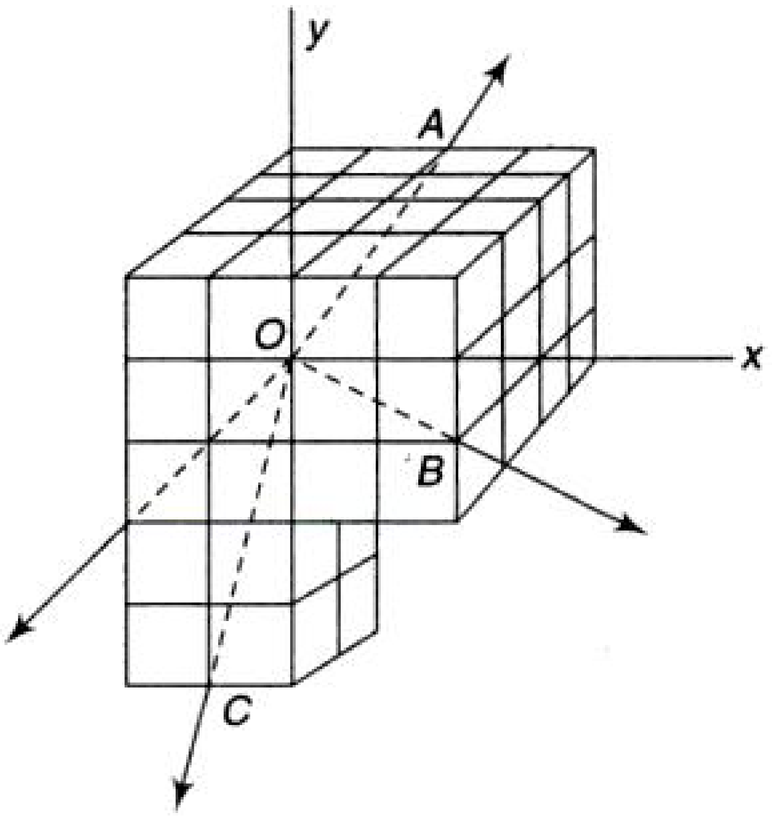


Fig. 5.4

Solution: We begin by computing the lengths of the lines OA, OB and OC, from the observed coordinates of the points A, B and C, as follows:

$$OA = \sqrt{(2)^2 + (3)^2 + 0} = \sqrt{13}$$

$$OA = \sqrt{(4)^2 + (1)^2 + (4)^2} = \sqrt{33}$$

$$OA = \sqrt{(1)^2 + (-2)^2 + (4)^2} = \sqrt{21}$$

Using these values, the direction cosines of the lines of action of the forces are easily computed and tabulated as shown in the table. Then using Eq. (b), we find the projections of the given forces as shown in the last three columns of the table.

Finally, making the summations of projections and using Eqs (c) and (d), we obtain

F_i	lb	$\cos \alpha_i$	$\cos \beta_i$	$\cos \gamma_i$	X_i	Y_i	Z_i
F_1	40	0.555	0.822	0	22.2	33.3	0
F_2	10	0.696	0.174	0.696	7.0	1.7	7.0
F_3	30	0.218	−0.436	0.873	6.5	−13.1	26.2
Σ					35.7	21.9	33.2

$$R = \sqrt{(158.9)^2 + (97.60)^2 + (147.60)^2} = 237.83 \text{ N}$$

$$\alpha = \cos^{-1} \frac{158.9}{237.8} = 48°04'$$

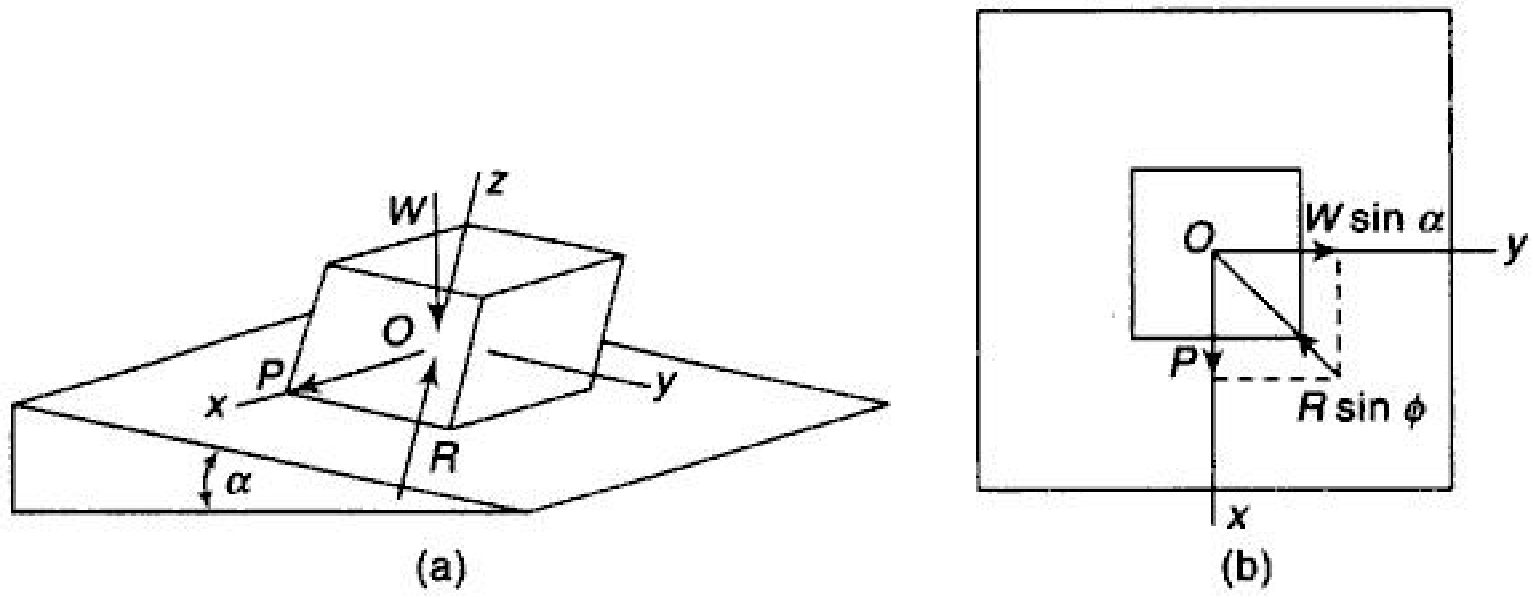


Fig. 5.8

Solution: At the instant of impending, slipping the block is in equilibrium under the action of the gravity force W, the applied force P, and a reaction R exerted by the inclined plane. Since these three forces are in equilibrium, we conclude that they must intersect in one point and also lie in one plane. Further, when sliding of the block impends, we know that the reaction R is inclined to the normal to be inclined plane by the angle of friction φ.

Choosing O as the origin of coordinates with the z-axis normal to the inclined plane and the x-axis horizontal and equating to zero the algebraic sum of the projections of all forces on the z-axis, we obtain

$$R \cos \varphi = W \cos \alpha \tag{f}$$

Projecting the entire system of forces onto the inclined plane, we obtain the system of coplanar forces in equilibrium as shown in Fig. 5.8(b), from which we conclude that

$$R^2 \sin^2 \varphi = P^2 + W^2 \sin^2 \alpha \tag{g}$$

Eliminating R between Eqs. (f) and (g) gives

$$P = W\sqrt{\mu^2 \cos^2 a - \sin^2 \alpha} \tag{h}$$

where $\mu = \tan \varphi =$ coefficient of friction. It will be noted that for the limiting case where $\alpha = \varphi$, Eq. (h) gives $P = 0$. Also for the limiting case where $\alpha = 0$ and the inclined plane becomes a horizontal plane, Eq. (h) gives $P = \mu W$.

We see from the first limiting case that when all available friction is already being used to resist sliding of the block down the plane, then there is no resistance to lateral slipping. This explains, for example, why a rear-wheel-drive automobile can skid so freely from side to side when climbing a grade on wet or icy pavement. For the same reason, a car loses lateral stability if the brakes are too suddenly applied so as to cause the tires to slip.

6. The small pulley in Fig. 5.9(a) drives the large one in a counter-clockwise direction by a V belt overrunning their rims, as shown. The angle of the V in both pulleys is 2α, the total angle of contact on the small pulley is β, and the coefficient of friction between the belt and rim is μ. Find the ratio S_1/S_2 between the tensions in the two branches of the belt when slipping impends

4. A mast *AB* supported by spherical socket at *A* and horizontal guy wires *BC* and *BD* carries a vertical load *P* at *B* as shown in Fig. B. Find the axial force induced in each of the three members of this system.(*Ans.* $S_1 = +0.8P$; $S_2 = +0.6P$; $S_3 = -1.4P$)
5. Repeat the solution of Prob. 4 if point *D* is 0.3 m vertically below the position shown in Fig. B and all other dimensions remain unchanged.
(*Ans.* $S_1 = +1.03P$; $S_2 = +0.75P$; $S_3 = -1.77P$)

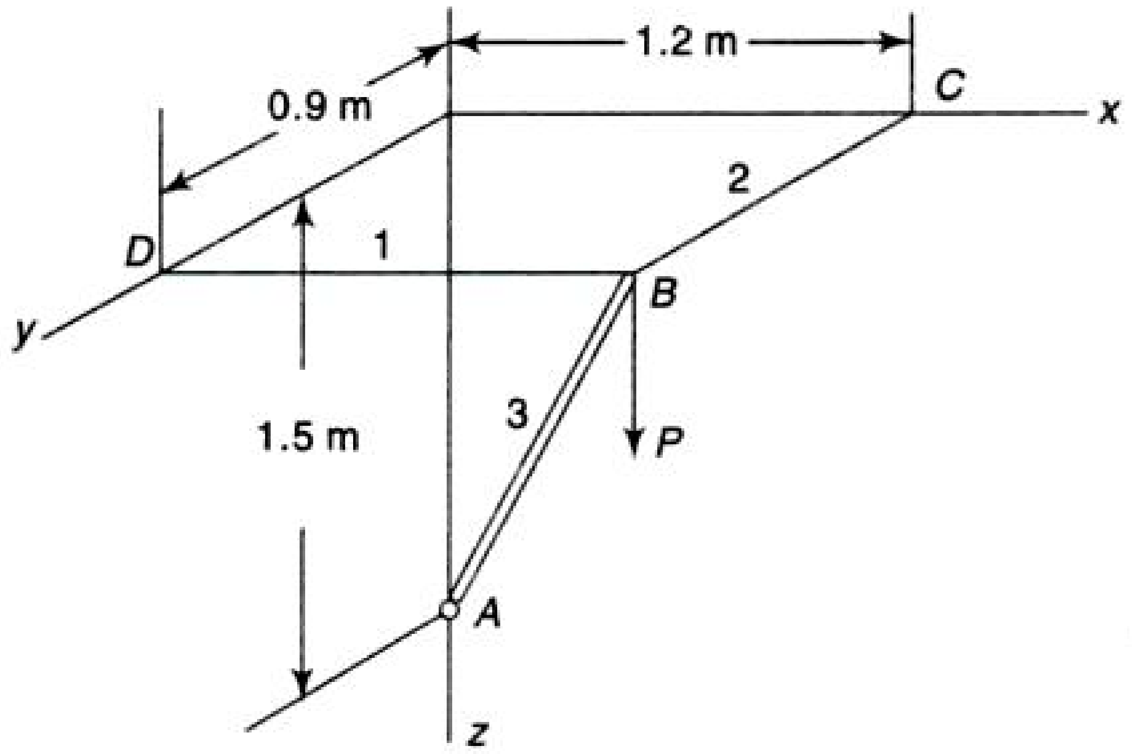


Fig. B

6. In the case of the tripod shown in Fig. C there is no friction between the ends of the legs and the floor on which they rest. To prevent slipping of the legs, their ends are connected by strings along the lines *AB*, *BC* and *AC*. Determine then the tensile force *S* in each of these strings if each leg makes 30° with the vertical and *P* is a vertical load. $\left(Ans.\, S + \frac{1}{2}P\right)$

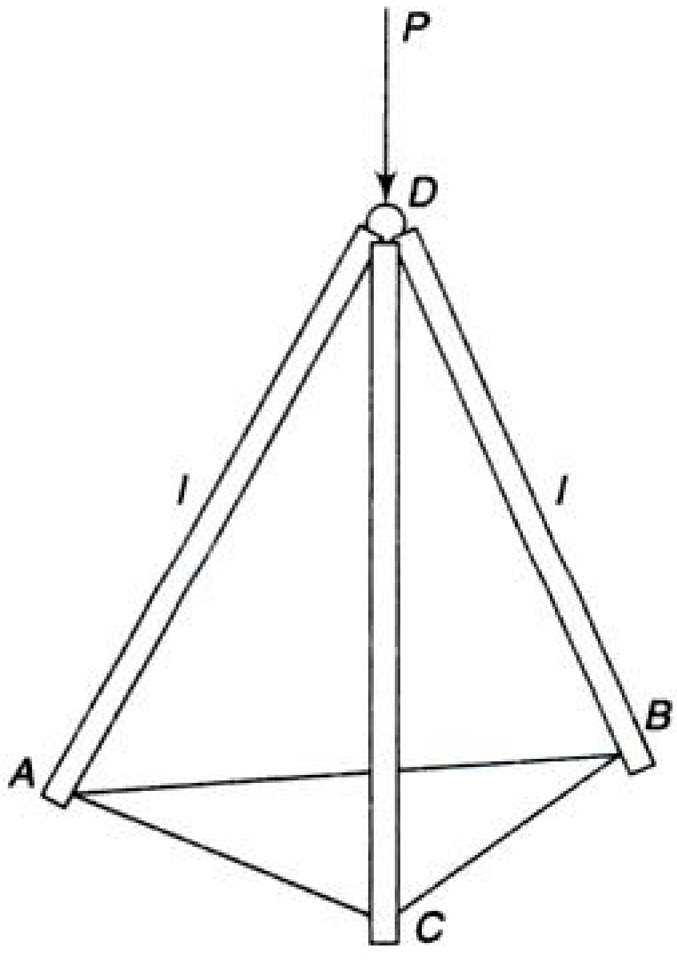


Fig. C

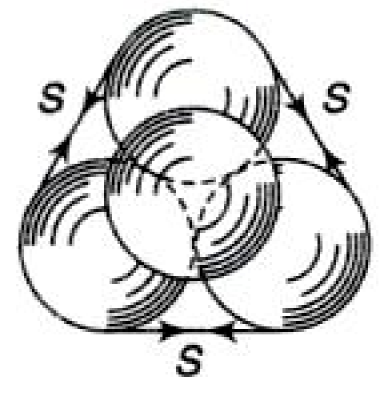


Fig. D

7. Four balls of equal radii *r* form a pyramid supported by a smooth horizontal plane surface (Fig. D). The three lower balls are held together by an encircling string as shown. Determine the tensile force *S* in this string if the weight of each ball is *Q* and the surfaces of the balls are perfectly smooth. Neglect any initial tension that may be in the string before the top ball is placed upon the other three.
(*Ans.* $S = 0.136\,Q$)

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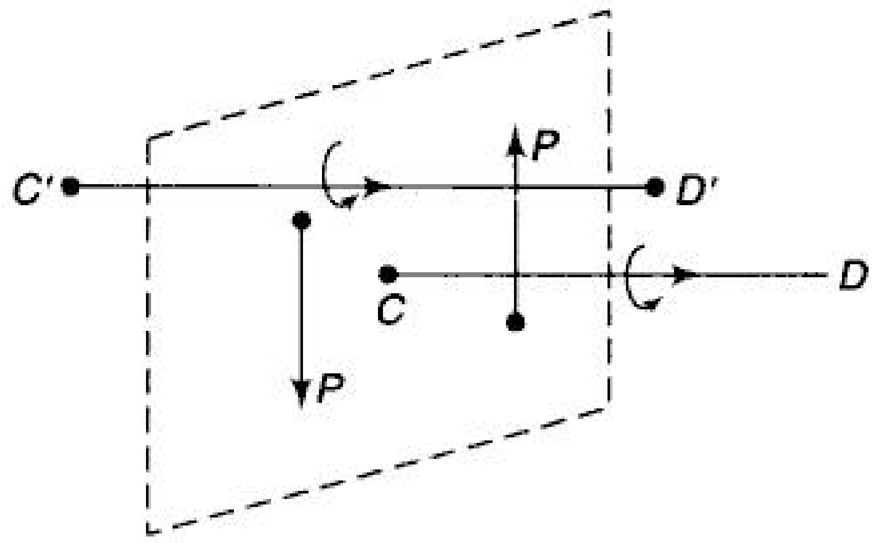

Fig. 5.17

couple can be displaced parallel to itself without changing the action of the couple, we conclude that any vector $\overline{CD}$ equal and parallel to the vector $\overline{C'D'}$ represents an equivalent couple. The vectorial representation of a couple is very useful when we have to deal with couples differently oriented in space, and in our further discussions a vector, such as $\overline{CD}$ or $\overline{C'D'}$, will be called a *moment vector* to distinguish it from a *force vector*.

Since the position of a couple in its plane is of no consequence and since the plane of a couple can be displaced parallel to itself without changing the action of the couple, it follows that any system of couples differently oriented in space can be represented by moment vectors that can be taken concurrent at any point in space. Hence it follows at once by analogy to the ease of concurrent forces in space that the resultant couple M of any system of couples $M_1, M_2, \dots M_n$ may be found by the method of projections.

Consider, for example, the system of couples in space as represented by the moment vectors $M_1, M_2, \dots, M_n$ in Fig. 5.18, where each couple must be visualized as acting anywhere in a plane normal to the corresponding moment vector. Taking coordinate axes x, y, z, as shown, and denoting by α_i, β_i, γ_i the direction angles of the vector M_i, we see that the projections of this vector are

$$\begin{aligned}(M_x)_i &= M_i \cos \alpha_i \\ (M_y)_i &= M_i \cos \beta_i \\ (M_z)_i &= M_i \cos \gamma_i\end{aligned} \tag{a}$$

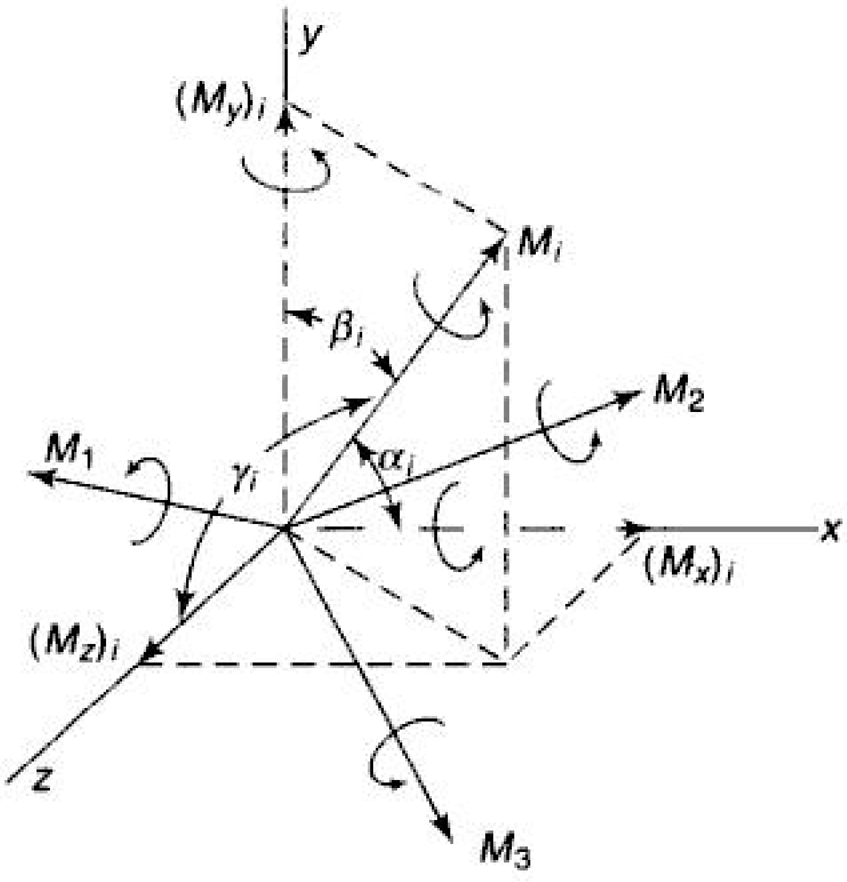

Fig. 5.18

Select the correct statement.

(a) Both A and R are true and R is the correct explanation of A.
(b) Both A and R are true and R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false and R is true.

[*Ans.* (a)]

2. A couple is completely defined by the following:
   I. The magnitude of its moment.
   II. The aspect of the plane in which it acts.
   III. The direction of rotation in the plane.
   (a) I is correct
   (b) I and II are correct
   (c) I and III are correct
   (d) I, II and III are correct

[*Ans.* (d)]

PROBLEM SET 5.3

1. Three circular disks *A*, *B* and *C* of radii $r_a = 375$ mm, $r_b = 250$ mm and $r_c = 125$ mm, respectively, are fastened at right angles to the ends of three rigidly connected arms which all lie in one plane as shown in Fig. A. If couples act on the disks *A* and *B* as shown in the figure, find the magnitude of the forces *P* of the couple that must be applied to the disk *C* and the angle α that the arm *OC* must make with the arm *OB* in order to have equilibrium. (*Ans.* $P = 222.5$ N, $\alpha = 143°08'$)

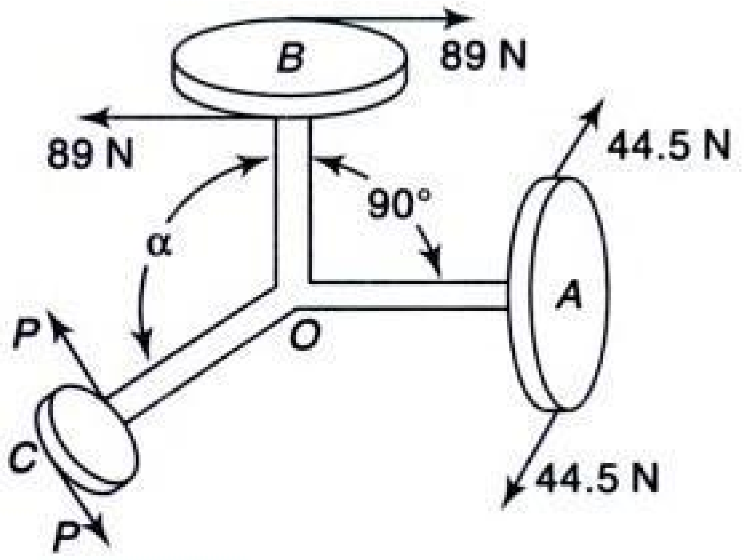

Fig. A

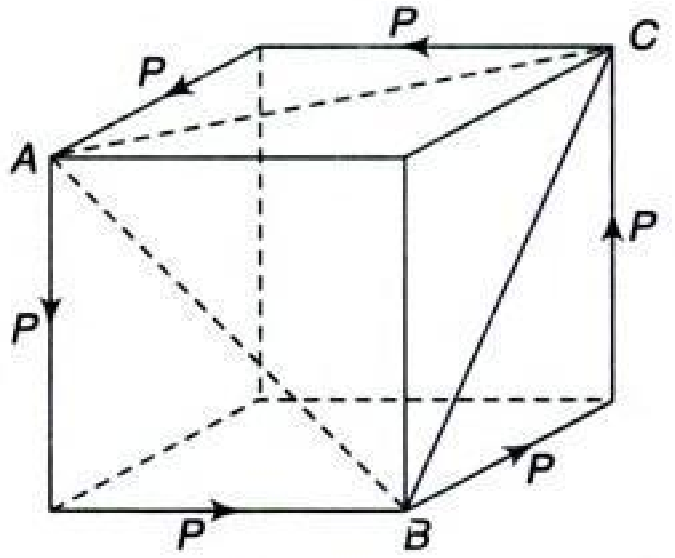

Fig. B

2. Six equal forces $P = 22.25$ N act on a cube with edges of length $a = 125$ mm as shown in Fig. B. Determine the resultant of this system of couples. (*Ans.* $M = 5.6$ N m in the octahedral plane *ABC*)

3. Couples of moments M_1, M_2, M_3 act on a prismatic block as shown in Fig. C. What must be the relationship between the magnitudes of moment of these couples if the block is in equilibrium? $\left(Ans. \dfrac{M_1}{\sin\alpha} = \dfrac{M_2}{\sin\beta} = \dfrac{M_3}{\sin\gamma}\right)$

4. A piece of round pipe in the form of a circular quadrant of radius *r* is attached to two mutually perpendicular vertical wall by flanges *A* and *B* as shown in Fig. D. If the pipe is subjected to uniformly distributed twisting moment of intensity *m* as shown, find the reactions at *A* and *B*. Assume that the ends of the pipe can rotate freely inside the flanges so that no twist can be exerted oneither flange. (*Ans.* $M_a = M_b = mr$)

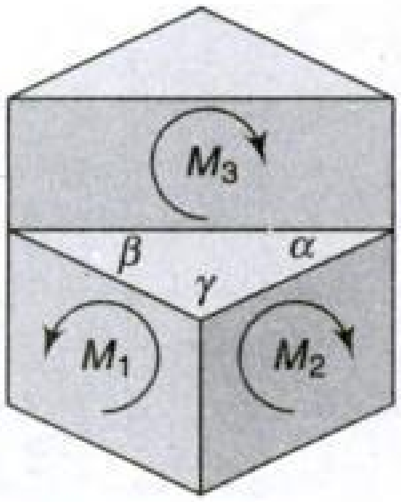

Fig. C

Solution: Neglecting friction, the bearing reactions will be vertical and we obtain a system of parallel forces in space that are in equilibrium. With B as origin and coordinate axes directed as shown, Eq. (26) become

$$R_a + R_b - P - S = 0$$
$$-Pc - R_o d + S(c + d) = 0$$
$$Pa - Sb = 0$$

From the last equation, we see that $S = Pa/b$. Substituting this in the other two equations and solving for R_a and R_b, we find

$$R_a = P\frac{a}{b}\left[1 + \frac{c}{d}\left(1 - \frac{b}{a}\right)\right] \qquad \text{(b)}$$

$$R_b = P\frac{a}{b}\left[1 + \frac{c}{d}\left(1 - \frac{a}{b}\right)\right]$$

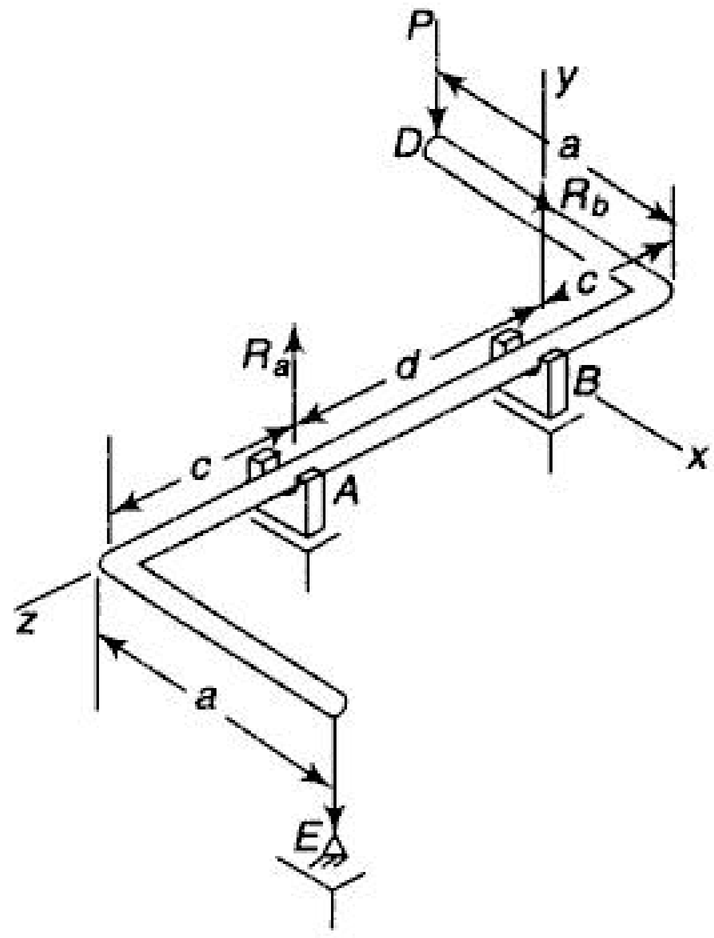

Fig. 5.24

Numerical values of R_a and R_b for given dimensions of the system can now be calculated from these formulas. We note, since the bearings are open, that negative values of R_o and R_b cannot be produced. A study of expression (b) shows that the reactions will both the positive, i.e. up, if the quantities within the brackets are positive. This requires

$$\frac{c+d}{c} > \frac{a}{b} > \frac{c}{c+d} \qquad \text{(c)}$$

Geometrically conditions (c) mean that the line CD joining the two ends of the shaft must cut the central portion of the shaft somewhere between A and B.

2. A movable crane mounted on three wheels, the bearings of which form an equilateral triangle ABC with sides of length a, rests on a horizontal track as shown in Fig. 5.25. The distribution of the weight Q of the crane itself is such that its center of gravity E is vertically above the centroid of the equilateral triangle ABC. For all possible values of the angle α that the vertical plane of the boom can make with the vertical middle plane of the crane, determine the corresponding maximum values of the load P that can be suspended from point D without causing the crane to tip from the track.

   Solution: For small values of the angle α, as indicated by the position FD of the boom [Fig. 5.25(b)], the condition of impending tipping of the crane about the axis AB with the wheel C lifting from the track will represent the criterion for the determination of the critical value of the load P, while for larger values of the angle α, as indicated by the position FD' of the boom, the condition of impending tipping of the crane about the axis AC with the wheel B lifting from the track will present the criterion for the determination of the critical value of the load P. Hence, the possibility of tipping of the crane about each of these axes must be investigated separately and the

Important Formulae

1. The resultant force may be defined analytically by the equations:

$$Z = \Sigma Z_i, \quad \bar{x} = \frac{\Sigma(Z_i x_i)}{\Sigma Z_i}, \quad \bar{y} = \frac{\Sigma(Z_i y_i)}{\Sigma Z_i}$$

2. The conditions that the system reduce to a resultant couple may be expressed analytically as follows:

$$\Sigma Z_i = 0, \quad M_x = \Sigma(Z_i y_i), \quad M_y = \Sigma(Z_i x_i)$$

3. The conditions of equilibrium for any system of parallel forces in space are expressed by the equations

$$\Sigma Z_i = 0, \quad \Sigma(Z_i y_i) = 0, \quad \Sigma(Z_i x_i) = 0$$

PRACTICE SET 5.4

Review Questions

1. What are the possibilities of reducing a parallel force system in space?
2. Write the equilibrium equations for a parallel force system in space.
3. How will you determine the resultant force of the parallel force system, if resultant force exists?
4. What is meant by statically indeterminate?

Objective Questions

1. The simplest resultant of a spatial parallel force system is always
   (a) a wrench
   (b) a resultant force
   (c) a resultant couple
   (d) a resultant force and a resultant couple

[*Ans.* (c)]

PROBLEM SET 5.4

1. A homogenous rectangular plate *ABCD* of width *a*, length *b*, and weight *Q* is supported horizontally by three vertical strings, as shown in Fig. A. Determine the axial forces S_1, S_2 and S_3 in the three supporting strings.

(*Ans.* $S_1 = S_2 = +Q/2$; $S_2 = 0$)

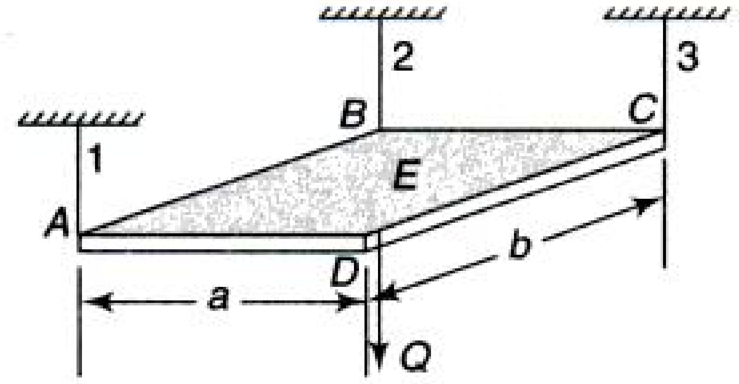

Fig. A

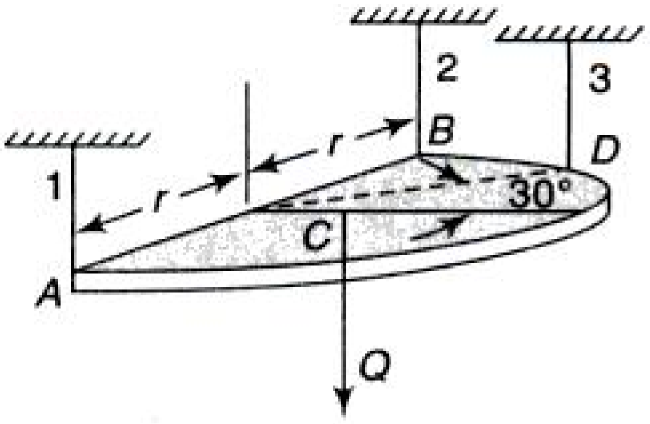

Fig. B

2. A homogenous semicircular plate of weight Q and radius r is supported in a horizontal plane by three vertical strings as shown in Fig. B. Determine the tensile forces S_1, S_2 and S_3 in these strings. (*Ans.* $S_1 = 0.38Q$; $S_2 = 0.13Q$; $S_3 = 0.49Q$)

with respect to an axis in this plane that is perpendicular to the direction of the forces and using the above statement regarding moments, we obtain for the perpendicular distance from the yz-plane to the line of action of the resistant (and consequently to the center of parallel forces) the following expression:

$$x_c = \frac{\sum (F_i x_i)}{\sum F_i} \tag{27a}$$

Next, we imagine the parallel lines of action of the given forces to be parallel to the xz-plane. Then proceeding as before, the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$y_c = \frac{\sum (F_i y_i)}{\sum F_i} \tag{27b}$$

Finally, we imagine the given parallel forces to act through their points of application parallel to the xy-plane. Then the perpendicular distance from this plane to the line of action of the resultant (and consequently to the center of parallel forces) is given by the equation

$$z_c = \frac{\sum (F_i z_i)}{\sum F_i} \tag{27c}$$

Since we know that the center of parallel forces is independent of the direction of the forces, we conclude that Eq. (27) define the coordinates of this point for any direction of the forces.

Since the center of gravity of a body (see Section 3.3) is the center of parallel gravity forces represented by the weights of the various particles of the body, it follows that the coordinates of the center of gravity of any body can be determined by the use of Eq. (27). In the case of a body of homogeneous material, we conclude that the position of the center of gravity depends only upon the shape of the body and not upon its density. Thus the center of gravity of a body of uniform density is coincident with the centroid of the volume of space occupied by the body.

It follows from the form of Eq. (27) that the center of gravity of a body of uniform density which has a plane of symmetry lies in that plane. If the body has two planes of symmetry, the center of gravity lies on the line of intersection of these planes. If the body has three planes of symmetry, the center of gravity lies at the point of intersection of these planes and is completely determined. Thus the center of gravity of a sphere of uniform density lies at the center of the sphere, the center of gravity of a right circular cylinder of uniform density lies at the midpoint of its geometric axis, the center of gravity of a cone of uniform density lies somewhere on its geometric axis, etc.

If a body may be considered as made up of several finite parts, the centers of gravity of which, individually, are known, then to locate the center of gravity of the composite body, it is only necessary to determine the coordinates of the center of parallel forces represented by the weights of the several parts applied, respectively, at the known centers of gravity of these parts.

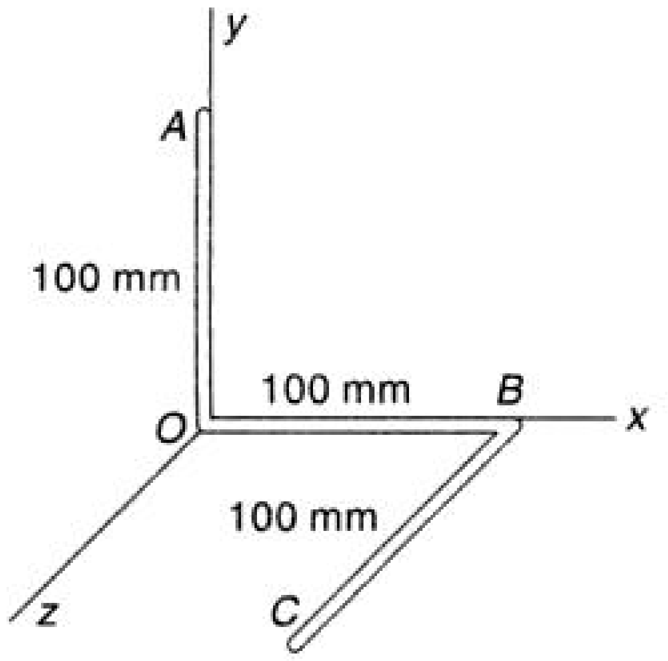

Fig. A

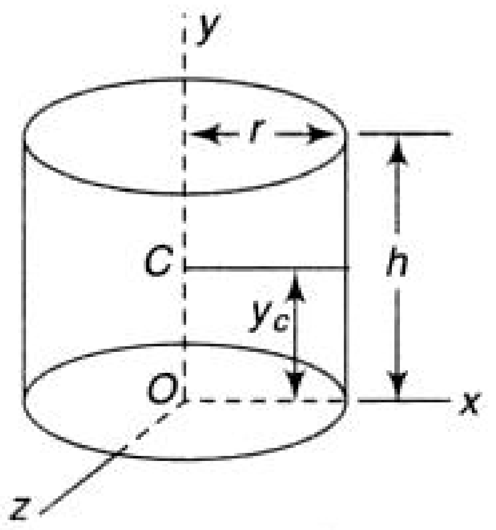

Fig. B

2. Determine the coordinate y_c of the center of gravity C of a right circular cylindrical can of height h and radius of base r if it is made of very thin metal of uniform thickness and density (Fig. B). The can is closed at the bottom and open at the top. (*Ans.* $y_c = h^2/(2h + r)$)
3. A steel shaft of circular cross-section has a circular steel hub pressed onto it as shown in Fig. C. For the dimensions shown in the figure, determine the distance x_c from the left end of the shaft to the center of gravity C of the composite body. (*Ans.* $x_c = 157$ mm)

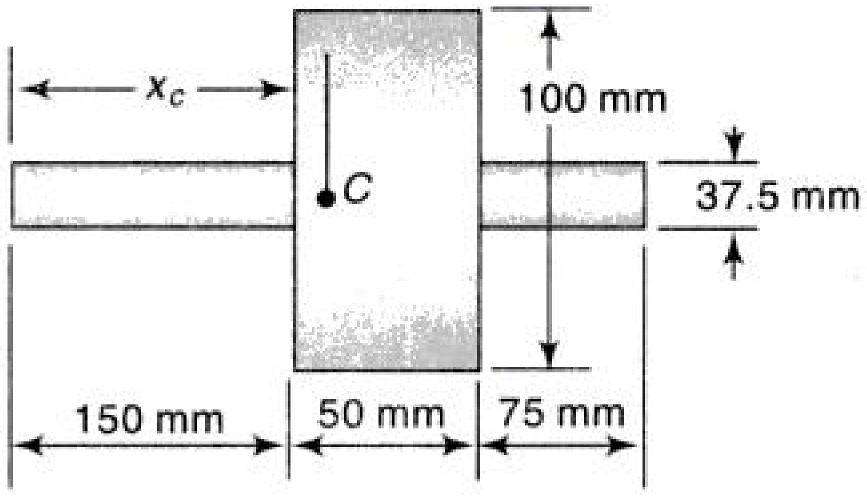

Fig. C

4. The corner of a rectangular box made of this sheet metal of uniform thickness and density is cut-off as shown in Fig. D. Determine the coordinates x_c, y_c and z_c of the center of gravity of the corner. (*Ans.* $x_c = 31.25$ mm, $y_c = 31.25$ mm; $z_c = 56.25$ mm)
5. Referring to Fig. E, locate the centroid of the composite area consisting of square in the xy-plane, a triangle in the yz-plane, and a circular quadrant in the xz-plane. (*Ans.* $x_c = 0.365a$; $y_c = 0.292a$; $z_c = 0.219a$)
6. Prove that the center of gravity of any homogeneous pyramid with base area A and altitude h lies on the line joining the vertex of the pyramid with the centroid of the area of its base at a distance equal to one-quarter of the altitude from the plane of the base.
7. Determine the height z_c of the center of gravity of a right circular cone above the plane of the base if the density of the material at each point in the cone is proportional to the distance of that point from the plane of the base.

$$\left(Ans.\, x_c = \frac{3}{4}h\right)$$

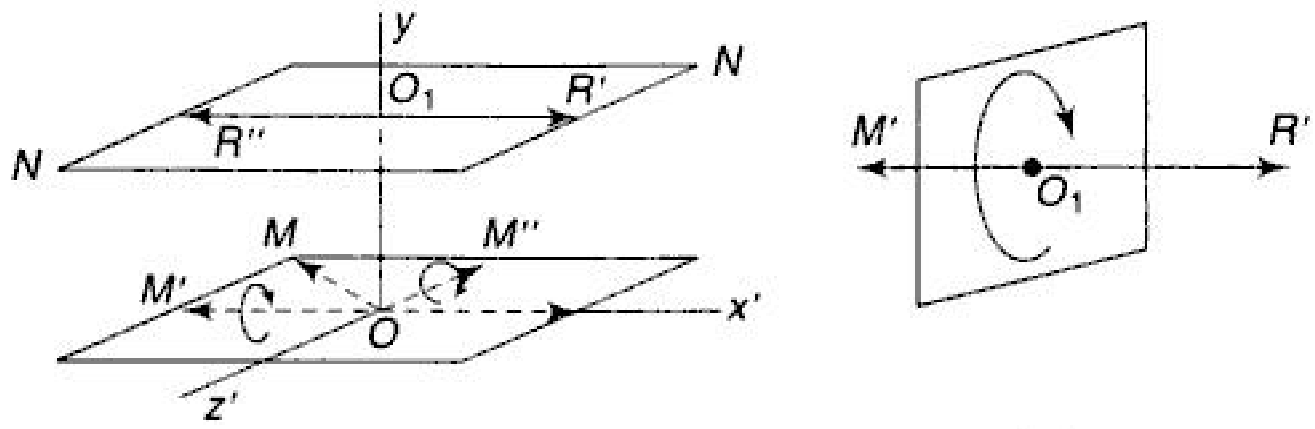

Fig. 5.31

(a) (b)

In this plane, we can always resolve the moment vector $\bar{M}$ into two rectangular components: M', coinciding with the line of action of R, and M'' perpendicular there to, as shown in the figure. Now at point O_1 on the y'-axis, we may introduce two oppositely directed collinear forces R' and R'' each equal and parallel to R without altering the action of the system. If we choose the distance $OO_1 = M''/R$, we see that the original force R at O and the force R'' at O_1 constitute a couple which balances the component M'' of the original couple M. Thus we have left only a force R' applied at O_1 and a couple M' in a plane normal to the line of action R'. The moment vector $\bar{M}'$ can now be moved to point O_1, and we have finally the simplified system shown in Fig. 5.31(b), where the resultant couple acts in a plane normal to the line of action of the resultant force. This simplest possible representation of a system of forces in space is sometimes called a *trench*, and *point* O_1 is called the *true center* of the system.

Equations of Equilibrium

From the preceding discussion, we conclude that, in the general case of a system of forces in space, equilibrium can exist only if both the resultant force R and the resultant couple M vanish. Thus the equations of equilibrium, as already obtained in Sections 5.1 and 5.3, are as follows:

$$\Sigma X_i = 0, \qquad \Sigma Y_i = 0, \qquad \Sigma Z_i = 0$$

$$\Sigma (M_x)_i = 0, \qquad \Sigma (M_y)_i = 0, \qquad \Sigma (M_z)_i = 0 \tag{28}$$

These six equations of equilibrium apply to any system of forces, and all cases discussed previously can be obtained from this general case.

If the forces are all parallel and we take the z-axis parallel to them, the first, second, and last equations will always be satisfied and we arrive at Eq. (26) of Section 5.4.

If the forces all intersect in one point and we choose this point as the origin of coordinates, the last three of Eq. (28) will always be satisfied and we arrive at Eq. (20) of Section 5.1.

If the forces are all in one plane which we can take as the xy-plane, the third, fourth, and fifth of Eq. (28) will always be satisfied and we arrive at Eq. (18) of Section 4.2.

In a similar manner, it can be shown that for the cases of parallel and concurrent forces in a plane, Eq. (28) can be reduced to Eq. (11) of Section 3.2 and Eq. (3) of Section 2.4, respectively.

We see from Eq. (28) that there are only six independent conditions of equilibrium for the general case of a system of forces in space. This means that in deal-

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the virtual displacement of points A and B are $a\delta\theta$ and $b\delta\theta$, perpendicular to the axis of the bar and oppositely directed.

In Fig. 6.2, we have a system of three rigid bodies consisting of crank, connecting rod and piston of an engine. The crank can rotate freely about the z-axis perpendicular to the plane of the figure through O and the configuration of the system is completely defined by the angle θ that OA makes with the fixed x-axis.

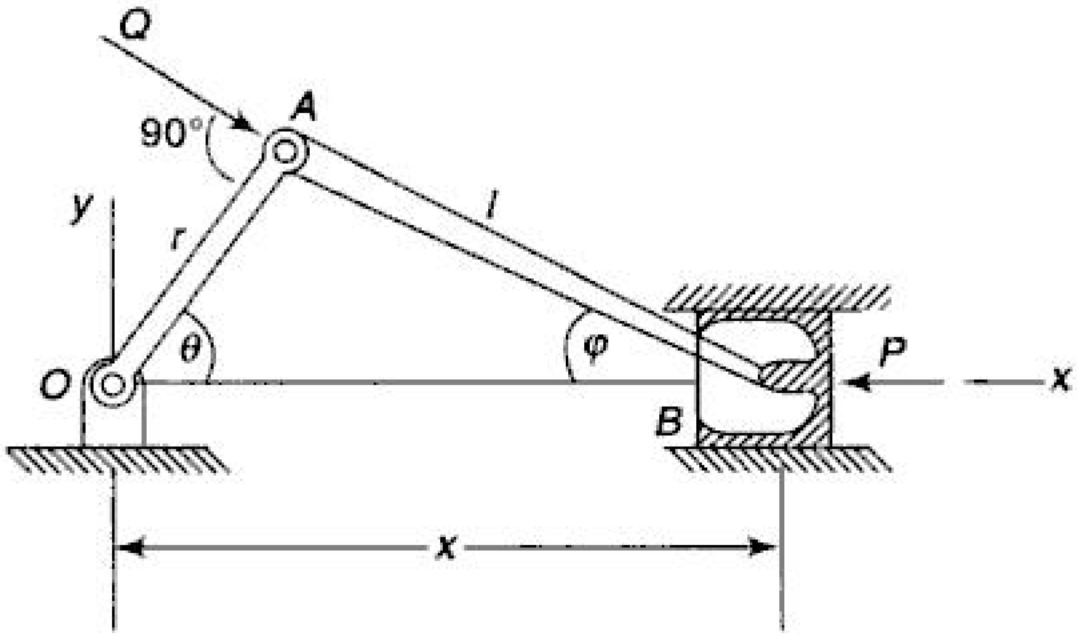

Fig. 6.2

Thus again we have a system with one degree of freedom and the angle θ can be taken as the coordinate of the system. To define virtual displacements of various points in this case, we give to the coordinate θ an infinitesimal increment $\delta\theta$. Then the virtual displacement of the crankpin A is $r\ \delta\theta$ perpendicular to OA. Having the displacement of point A and observing that point B is constrained to move along the x axis, we can now express the corresponding virtual displacement of B as follows: The distance x of the piston from O is

$$x = r\cos\theta + l\cos\varphi$$

Then the change in x due to an increase $\delta\theta$ in the angle θ is

$$\delta x = -r\sin\theta\delta\theta - l\sin\varphi\delta\varphi \tag{a}$$

From ΔAOB (Fig. 6.2), we have

$$\sin\varphi = \frac{r}{l}\sin\theta, \qquad \varphi = \arcsin\left(\frac{r}{l}\sin\theta\right)$$

Substituting these expression into Eq. (a), we obtain for the virtual displacement of the piston

$$\delta x = -r\ \delta\theta\left[1 + \frac{\cos\theta}{\sqrt{(l/r)^2 - \sin^2\theta}}\right]\sin\theta \tag{b}$$

If, for example, we take $\theta = 60°$ and $l/r = 2$, the virtual displacements of A and B will be $r\ \delta\theta$ and $-1.11r\ \delta\theta$, respectively. From these two examples, we see that the problem of defining virtual displacements of various points of a movable system is purely one of geometry.

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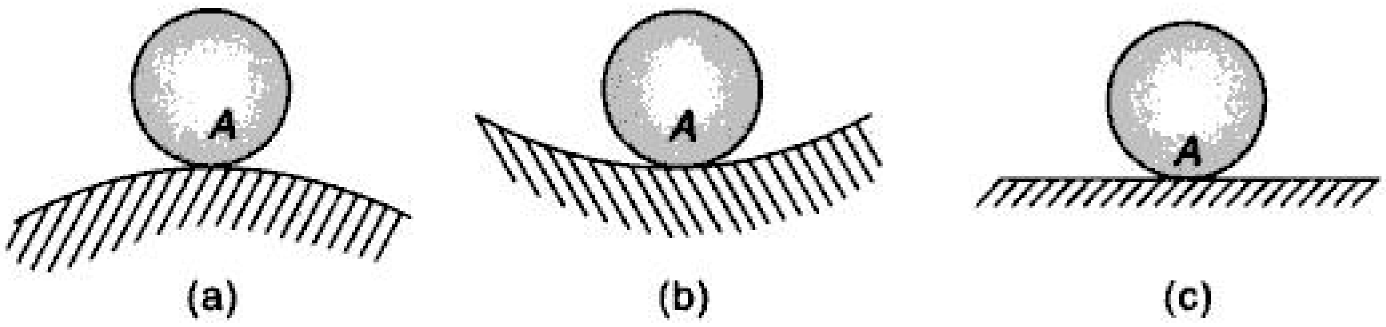

Fig. 6.14

this position, it tends to return back to it as soon as the force is removed. In Fig. 6.14(c), the ball is placed on a smooth horizontal plane, and in this case it is evident that it is in equilibrium in any position and we have *indifferent* equilibrium.

To establish, by using the principle of virtual work, that equilibrium exists in each of the three cases represented in Fig. 6.14, we give to the ball in each case a virtual displacement, i.e. an infinitesimal displacement in which the ball remains in contact with the supporting surface. For such a displacement, we assume that the point of contact of the ball moves horizontally in the plane tangent to the supporting surface at the point of contact. Then the center of gravity of the ball moves also horizontally and the gravity force W does not produce work. Thus we conclude that in all three cases represented in Fig. 6.14 there is a condition of equilibrium.

To decide if each of the above positions of equilibrium is stable or unstable, a more refined calculation of the work done by the active forces on a virtual displacement of the ball is required. Taking the case shown in Fig. 6.14(a) and considering an infinitesimal displacement $AA_1 = \delta s = r\delta\varphi$, we see from Fig. 6.15 that such a displacement involves lowering of the ball by the amount

$$AB = r(1 - \cos \delta\varphi) \approx \frac{r}{2}(\delta\varphi)^2$$

which is a small quantity of the second order if the displacement δs is considered as a small quantity of the first order. It is not necessary to consider this small quantity of higher order in deciding the question of whether or not equilibrium exists. But if the question arises regarding the kind of equilibrium, i.e. whether it is stable or unstable, this second-order quantity must be taken into account. Thus we conclude that, in the case of an unstable equilibrium as represented in Fig. 6.14(a), the gravity force of the ball produces on any virtual displacement a positive work and as a result of the slightest displacement, a reactive force in the direction of motion appears and the ball has the tendency to move away from its position of equilibrium. In the same manner it can be shown that in the case of a stable equilibrium, as represented in Fig. 6.14(b), there will be negative work done by the gravity force on any virtual displacement of the ball. Thus if by some accidental force, the ball is slightly displaced from its position of equilibrium, a reactive force opposing this displacement appears and there is a tendency for the ball to return to its initial position.

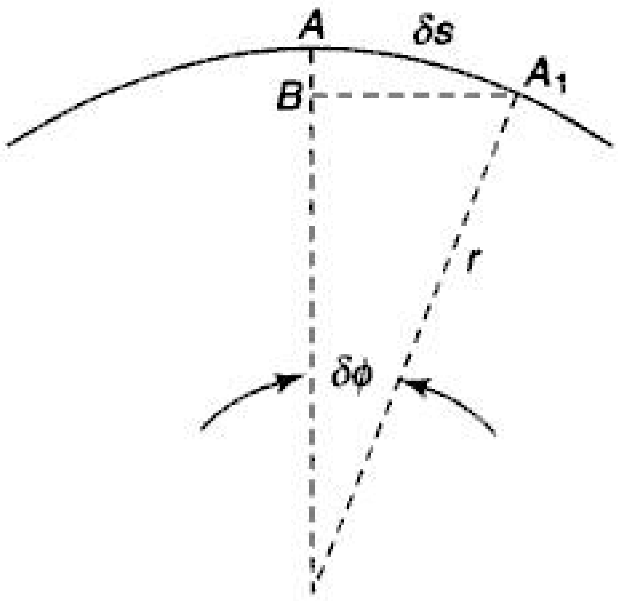

Fig. 6.15

as shown in the figure. Neglecting friction and the thickness of the bar *AB*, establish the criterion of stability of the system in the configuration shown.

(*Ans.* $Wl/2 < Qa\,[(h-a)/a)^2$)

9. A hemispherical cup of radius *r* and having its center of gravity at *C* rests on top of a spherical surface of radius *R* as shown in Fig. I. Assuming that there is sufficient friction to prevent slipping, establish the criterion of stability of the cup in the position shown.

$$\left(Ans.\ R > r\left[\left(\frac{r}{c}\right)-1\right]\right)$$

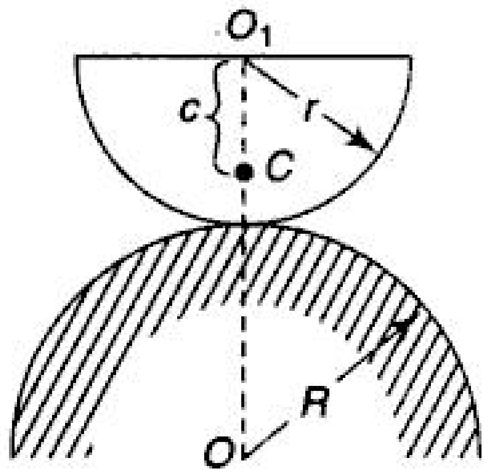

Fig. I

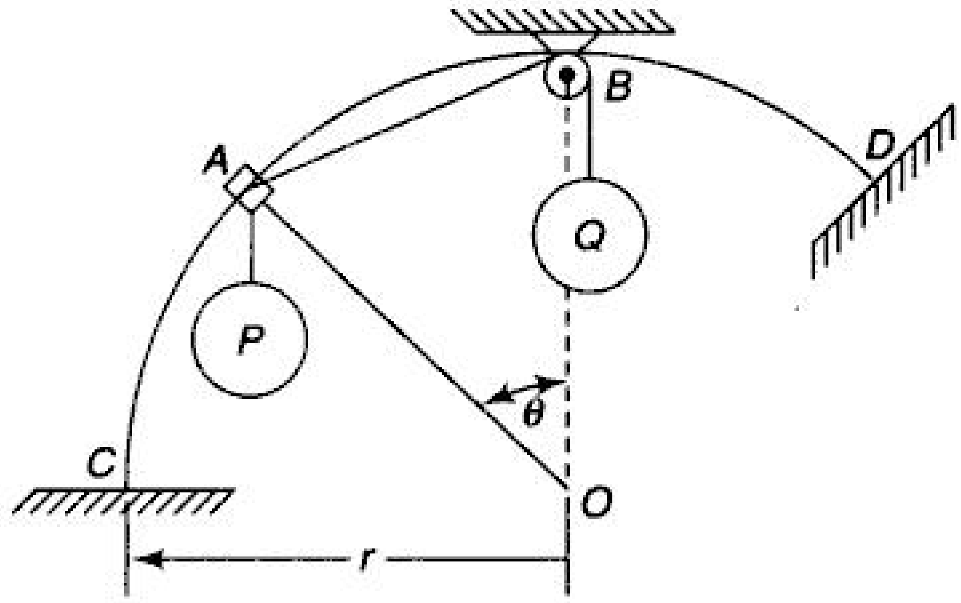

Fig. J

10. Using the principle of virtual work and taking θ coordinate, determine all possible configurations of equilibrium of the system in Fig. J and investigate the stability of each. The bead *A* can slide freely on the circular wire *CD* and the pulley at *B* is negligibly small.

2. From the top of a tower of height $h = 36$ m, a ball is dropped at the same instant that another is projected vertically upward from the ground with an initial velocity $v_0 = 18$ m/s. How far from the top do they pass and with what relative velocity?

   Solution: For the ball that is dropped, we choose the top of the tower as the origin and consider downward displacement as positive. Then this ball will have neither initial displacement nor initial velocity and from Eq. (37b), its displacement at any instant will be

   $$x_1 = \frac{1}{2}gt^2 \tag{f}$$

   For the other ball we choose the ground as the origin and consider upward displacement as positive. Then from Eq. (37a), its displacement at any instant will be

   $$x_2 = v_0t = \frac{1}{2}gt^2 \tag{g}$$

   When the ball pass, we must have

   $$x_1 + x_2 = h \tag{h}$$

   Substituting the values of x_1 and x_2 from Eqs (f) and (g) into Eq. (h), we obtain

   $$\frac{1}{2}gt^2 + v_0t - \frac{1}{2}gt^2 = h$$

   from which, using the given numerical data, $t = 2$ s. Using this value of t in Eq. (f), we find

   $$(x_1)_{t=2} = 19.32 \text{ m}$$

   Differentiating Eqs (f) and (g) once each with respect to time, we find that at the instant $t = 2$ s, the two balls are moving downward with velocities of 19.32 and 1.32 m/s, respectively. Hence the balls pass 19.32 m below the top of the tower with a relative velocity of 18 m/s, 2 s after starting.

3. A particle projected vertically upward is at a height h after t_1 s and again after t_2 s. Find this height h and also the initial velocity v_0 with which the particle was projected.

   Solution: Neglecting air resistance, the particle at all times is moving under the action of its own gravity force W which is always directed vertically downward. We take the x-axis along the vertical line of motion, the origin at the starting point, and consider upward displacement as positive. Then, from Eq. (37a), we have for the instant $t = t_1$:

   $$h = v_0t_1 - \frac{1}{2}gt_1^2 \tag{i}$$

   Likewise, for the instant $t = t_2$

   $$h = v_0t_2 - \frac{1}{2}gt_2^2 \tag{j}$$

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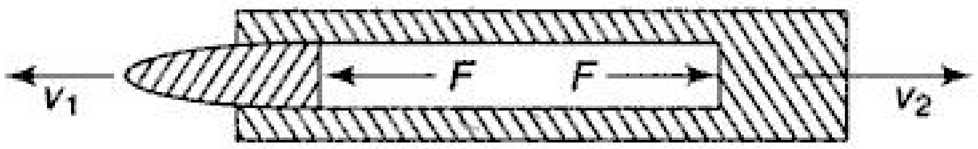
The equation of momentum and impulse is particularly useful when we are dealing with a system of particles, since in such case calculation of the impulse can often be eliminated. As a specific example, let us consider the case of a gun and shell as shown in Fig. 7.36, which may be considered as a system of two particles. During the extremely short interval of explosion, the forces F acting on the shell and gun and representing the gas pressure in the barrel are varying in an unknown manner and a calculation of the impulses of these forces would be extremely difficult. However, the relation between the velocity of the shell and the velocity of recoil of the gun can obtained without calculation of the impulse. Since the forces F are in the nature of action and reaction between the shell and gun, they must at all times be equal and opposite, and hence their impulses for the interval of explosion are equal and opposite, since the forces act exactly the same time t.

Fig. 7.36



Thus if W_1 and W_2 are the weights of the shell and gun, respectively, we find, assuming the initial velocities to be zero and neglecting all external forces that

$$\frac{W_1}{g} v_1 = \int F dt, \qquad \frac{W_2}{g} v_2 = \int F dt$$

Then for the entire system

$$\frac{W_1}{g} v_1 = \frac{W_2}{g} v_2$$

from which we obtain

$$\frac{v_1}{v_2} = \frac{W_2}{W_1}$$

We see that the velocities of the shell and gun after discharge are in opposite directions and inversely proportional to the corresponding weights.

We obtain a great simplification in the above example owing to the fact that no external forces act on the system but only internal forces in the nature of action and reaction. Internal forces in a system of particles always appear as pairs of equal and opposite forces and need not be considered when applying the equation of momentum and impulse. Thus we may state that, in the case of any system of particles to which no external forces are applied, the momentum of the system remains unchanged, since the total impulse is zero. This is sometimes called the principle of *conservation of momentum*.

Examples Examples Examples Examples Examples

1. A flat car can roll without resistance along a horizontal track as shown in Fig. 7.37. Initially, the car together with a man of weight w is moving to the right with speed v_0. What increment of velocity Δv will the car obtain if the

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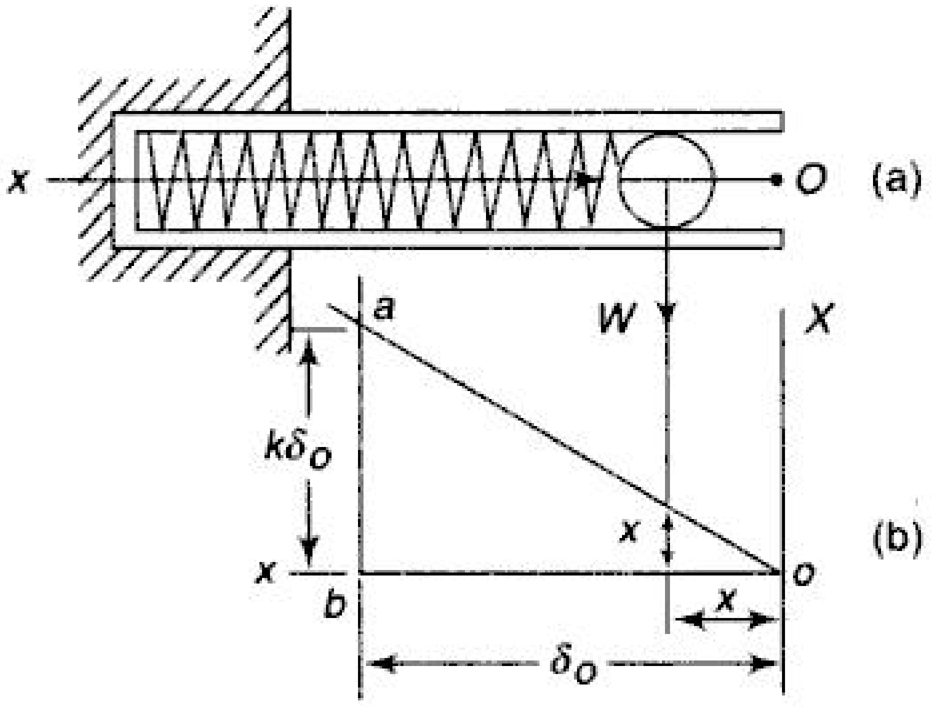

Fig. 7.40

Solution: Denoting by X the force exerted by the spring on the ball after release and during expansion of the spring, we have $X = kx$, where x is measured from the position of the ball corresponding to the unstressed length of the spring. The corresponding force–displacement diagram is shown in Fig. 7.40(b). We see that the initial force on the ball when $x = \delta_0$ is $k\delta_0$ and that it decreases uniformly to zero when $x = 0$. The corresponding work of the force X is represented by the area *oab* of the diagram and is $k\delta_0\ \delta_0/2 = k\delta_0^2/2$. Equation (47) then becomes

$$\frac{W}{g}\frac{\dot{x}^2}{2} = \frac{k\delta_0^2}{2}$$

from which

$$\dot{x} = \delta_0\sqrt{\frac{kg}{w}} \tag{f}$$

As would be expected, this is the same result obtained in Section 7.6 for the maximum velocity of a weight W attached to a spring of constant k and set in vibration by an initial displacement δ_0.

2. A weight W that can slide freely up and down a prismatic steel bar BC of length l and cross-sectional area A is allowed a free fall through the distance h (Fig. 7.41). Assuming the mass of the bar to be negligible compared with that of the falling weight, find the resulting dynamic elongation δ of the bar.
   Solution: The weight starts from rest, travels a total distance $h + \delta$, and again comes to rest after having stretched the bar an amount δ. Hence the net change in kinetic energy is zero, and we conclude from Eq. (47) that the net work of all forces acting through this displacement

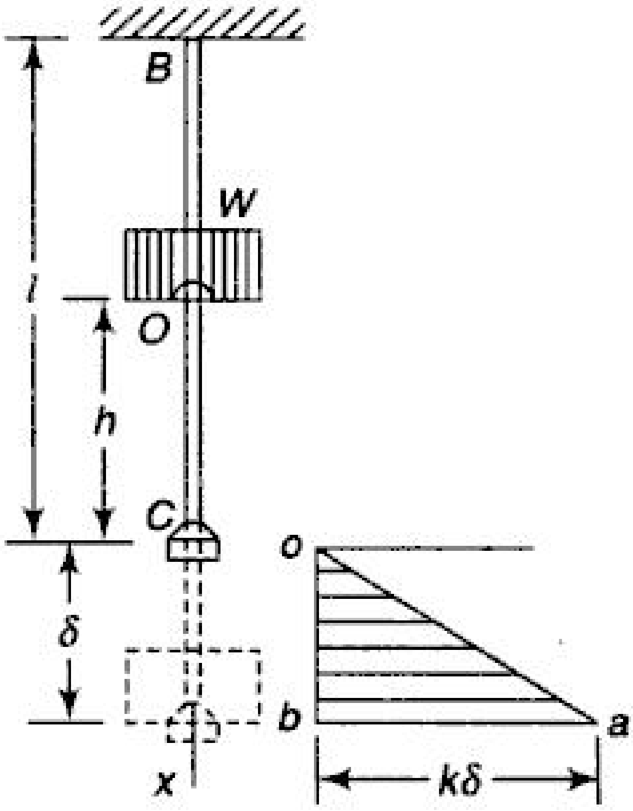

Fig. 7.41

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4. A V-tube having a uniform bore of cross-sectional area A stands with its two branches inclined to the vertical by equal angles a as shown in Fig. B. Calculate the period of oscillation of a column of liquid of total length l and specific weight w if initially displaced from its equilibrium position in the tube as shown in the figure.

(*Ans.* $\tau = 2\pi\sqrt{(l \sec \alpha)/2g}$)

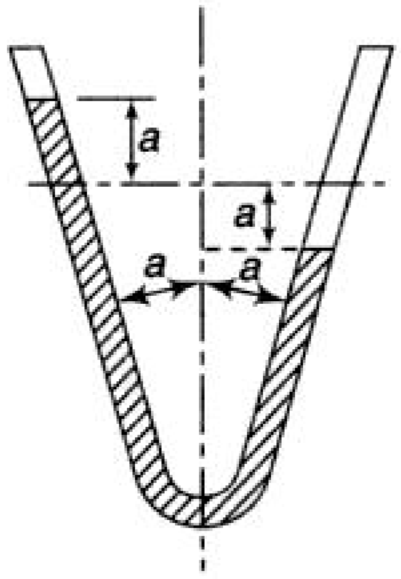


Fig. B

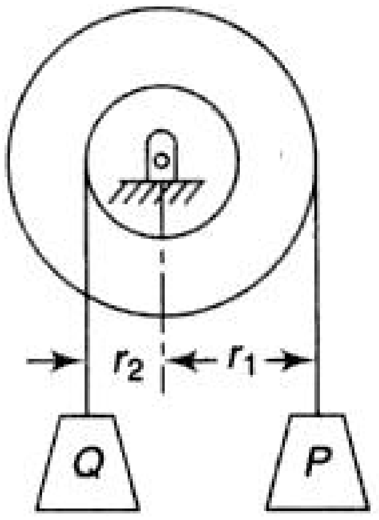


Fig. C

5. If the system in Fig. C is released from rest in the configuration shown, find the velocity v of the falling weight P as a function of its displacement x. Neglect friction and inertia of the pulleys and assume the following numerical data: $P = Q = 44.5$ N, $r_1 = 150$ mm, $r_2 = 100$ mm, $x = 3$ m. (*Ans.* $v = 3.66$ m/s)

6. The two blocks in Fig. D have weights $P = 44.5$ N, $Q = 22.5$ N and the coefficient of friction between the block P and the horizontal plane is $\mu = 0.25$. If the system is released from rest and the block Q falls a vertical distance $h = 0.6$ m, what velocity v will it acquire? Neglect friction in the pulley and extensibility of the string. (*Ans.* $v = 1.392$ m/s)

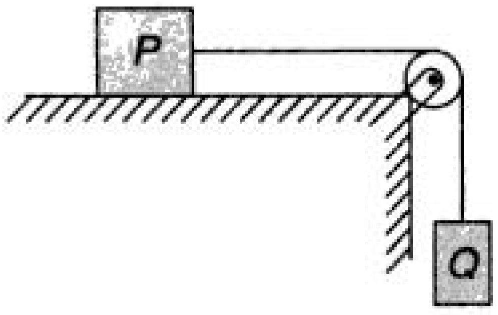


Fig. D

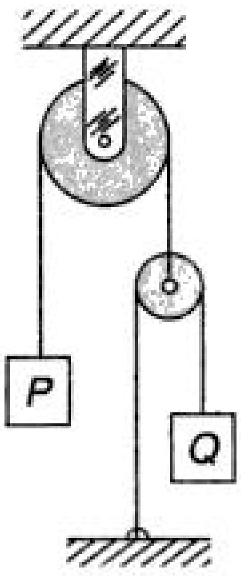


Fig. E

7. If the system in Fig. E is released from rest in the configuration shown, find the velocity v of the block Q after it falls a distance $h = 3$ m. Neglect friction and inertia of the pulleys and assume that $P = Q = 44.5$ N. (*Ans.* $v = 4.82$ m/s)

8. A length l of smooth straight pipe held with its axis inclined to the horizontal by an angle 30° contains a flexible chain also of length l. Neglecting friction and assuming that, after release, the chain falls vertically as it emerges from the open end of the pipe, find the velocity v with which it leaves the pipe. (*Ans.* $v = \sqrt{(3g/2}$)

we divide the second equation by the first and obtain

$$v_1 + v_1' = v_2 + v_2'$$

or

$$v_1' - v_2' = -(v_1 - v_2) \tag{c}$$

This equation represents a combination of the law of conservation of momentum and conservation of energy. It states that for an elastic impact the relative velocity after impact has the same magnitude as that before impact but with reversed sign. Using this idea in conjunction with that of conservation of momentum, we have for the case of elastic impact.

$$W_1 v_1 + W_2 v_2 = W_1 v_1' + W_2 v_2'$$

$$v_1' - v_2' = -(v_1 - v_2) \tag{51}$$

To illustrate the use of Eq. (51), let us consider several particulars cases. If, in Fig. 7.49, we have $W_1 = W_2$, Eq. (51) become

$$v_1 + v_2 + v_1' + v_2' \tag{d}$$

Subtracting and adding these equations, we find

$$v_1' = v_2 \qquad \text{and} \qquad v_2' = v_1$$

This shows that after an elastic impact, equal weights simply exchange velocities. If the weight W_2 was at rest before impact ($v_2 = 0$), Eq. (d) give

$$v_1' = 0 \qquad \text{and} \qquad v_2' = v_1$$

In this case, the striking ball simply stops after having imparted its velocity to the other ball. This phenomenon can be observed in the case of a moving billiard ball which squarely strikes one that was at rest.

Again, if the two balls were moving toward each other with equal speeds v before impact, an exchange of velocities will simply mean that they rebound from one another with the same speed with which they collided.

As another special case, we assume that $W_2 = \infty$ while W_1 remains finite and further $v_2 = 0$. This will represent the case of an elastic impact of a ball against a flat immovable obstruction, such as dropping a ball on a cement floor. Dividing the first of Eqs. (51) by W_2 we obtain $v_2' = 0$, as would be expected if W_2 is immovable. Then from the second equation, we find $v_1' = -v_1$. This shows that the striking ball rebounds with the same speed with which it hits the obstruction. It must be remembered that each of the examples discussed here assumes perfect elasticity so that no energy is lost during impact.

Semielastic Impact

Under actual conditions we must expect some deviation from perfect elasticity, and owing to this fact there always will be some loss in energy of the system during impact so that the relative velocity after impact is smaller than before and instead of Eq. (c) we must take

$$v_1' - v_2' = -e(v_1 - v_2) \tag{e}$$

where e is a numerical factor less than unity and is called the coefficient of restitution for the materials. Using this, we have for the general case of semi-elastic direct central impact, the following equations:

8

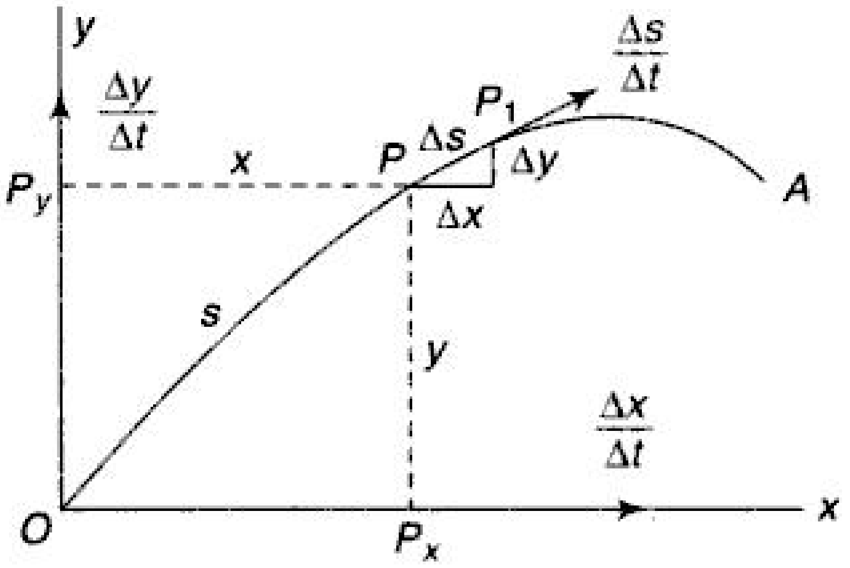
Curvilinear Translation

8.1 KINEMATICS OF CURVILINEAR MOTION

When a moving particle describes a curved path, it is said to have *curvilinear motion*. We shall now discuss the kinematics of such motion, assuming that the path of the particle is a plane curve.

Displacement

To define the position of a particle P in a plane (Fig. 8.1), we need two coordinates, x and y.



y
$\frac{\Delta y}{\Delta t}$
$\frac{\Delta s}{\Delta t}$
P_1
x
P
Δs
Δy
P_y
Δx
A
s
y
$\frac{\Delta x}{\Delta t}$
O
P_x
x

Fig. 8.1

As the particle moves, these coordinates change with time and we have the displacement–time equations

$$x = f_1(t), \qquad y = f_2(t) \tag{53}$$

When these two expressions are given, the motion of the particle in its plane is completely defined.

Instead of Eq. (53), we can also define the motion of a particle in a plane by the equations

$$y = f(x), \qquad s = f_1(t) \tag{53'}$$

3. Investigate the motions of the points A, B and C of a connecting rod (Fig. 8.7) if the crankpin A is moving with uniform speed v along the circle of radius r and the point B is constrained to follow the x-axis.

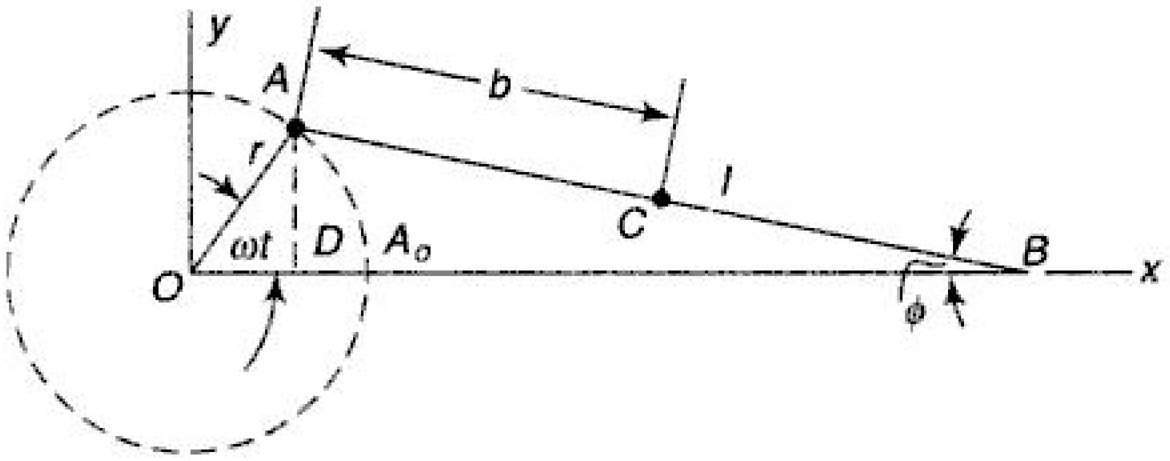


Fig. 8.7

Solution: We begin with the motion of point A, assuming that at the initial moment ($t = 0$) it has the position A_0. Then denoting by ω the angle of the arc which the point A describes in unit time, we have $\omega = v/r$, where r is the length of the crank. The angle A_0OA is then equal to ωt, and the coordinates of the point A are

$$x = r \cos \omega t, \qquad y = r \sin \omega t \tag{t}$$

The coordinate x of the point B, which, of course, has rectilinear motion, is obtained by projecting on the x-axis the length r of the crank and the length l of the connected rod. Then

$$x = r \cos \omega t + l \cos \varphi \tag{u}$$

Noting from the figure that

$$r \sin \omega t = l \sin \varphi$$

We obtain the following expression for $\cos \varphi$:

$$\cos \varphi = \sqrt{1 - \sin^2 \varphi} = \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t}$$

and substituting in Eq. (u) above, we have

$$x = r \cos \omega t + l \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \tag{v}$$

It is interesting to note that, while the projection D of the crankpin on the x-axis performs simple harmonic motion, the motion of the point B is more complicated.

For any point C on the axis of the connecting rod at the distance b from the crankpin A, we obtain

$$x = r \cos \omega t + b \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t}, \qquad y = \frac{l - b}{l} r \sin \omega t \tag{w}$$

In the particular case where $r = l$, we have from Eq. (w)

$$x = (r + b) \cos \omega t, \qquad y = (r - b) \sin \omega t \tag{x}$$

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Then since $v = ds/dt$, while for small values of θ we have $\sin\theta \approx \theta = s/l$, this equation reduces to

$$\frac{d^2s}{dt^2} + \frac{g}{l}s = 0 \tag{a}$$

This is the differential equation of simple harmonic motion [Eq. (40)] and the period is

$$\tau = 2\pi\sqrt{\frac{l}{g}} \tag{b}$$

3. A particle of weight W moves with uniform speed v along the cosine curve ABC in a vertical plane [Fig. 8.9(a)].

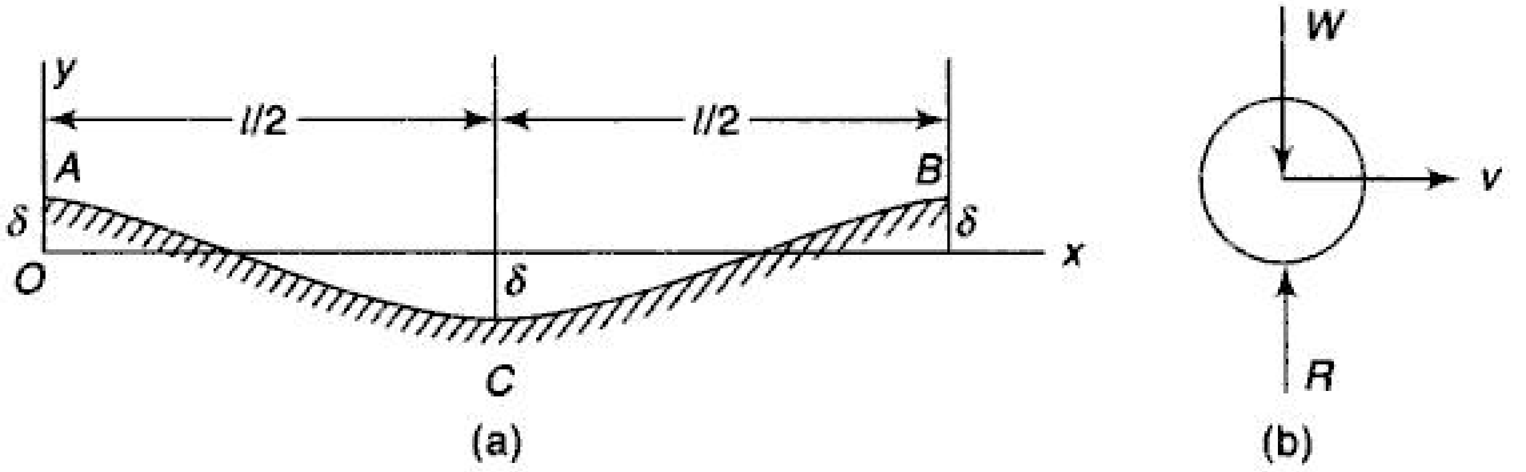

Fig. 8.9

The curve is defined by the equation

$$y = \delta\cos\frac{2\pi x}{l} \tag{c}$$

Find the presssure R exerted by the particle on the path as it passes the lowest point C.

Solution: Considering the particle at C [Fig. 8.9(b)] and using the second of Eq. (58), we have

$$\frac{W}{g}\frac{v^2}{\rho} = R - W$$

from which

$$R = W\left(1 + \frac{v^2}{g\rho}\right) \tag{d}$$

To find the curvature $1/\rho$, we use the known formula

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{e}$$

By differentiation of Eq. (c), we obtain

$$\frac{dy}{dx} - \frac{2\pi\delta}{l}\sin\frac{2\pi x}{l}$$

This distance is called the *range* of the projectile. If we replace $\dot{x}_0$ and $\dot{y}_0$ by their respective values $v_0 \cos \alpha$ and $v_0 \sin \alpha$, expression (i) defining the range becomes

$$r = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_2^0}{g} \sin 2\alpha \tag{i'}$$

It is seen that for a given initial velocity v_0 the maximum range is obtained when $a = 45°$ and this maximum range is

$$r_{\max} = \frac{v_2^0}{g} \tag{i''}$$

It must be remembered that all of the foregoing discussion of the motion of a projectile neglects the effects of air resistance, and for the speeds with which projectiles usually travel this factor is by no means negligible. Consideration of the effect of air resistance, however, greatly complicates the problem and is beyond the scope of this book.[2] In each of the following problems it will be assumed that the projectile moves without air resistance.

Examples Examples Examples Examples Examples

1. A cannon fires its projectile with such an initial velocity and at such an angle of elevation that the range is r and the maximum height to which the projectiles rises is h. Find the maximum range that can be obtained with the same initial velocity.

   Solution: Denoting by v_o and α the unknown initial velocity and angle of elevation and using Eqs. (h) and (i′), we may write

$$h = \frac{v_2^0}{g} \frac{\sin^2 \alpha}{2}, \qquad r = \frac{v_2^0}{g} \sin 2\alpha \tag{j}$$

   which, since the maximum range is v_0^2/g as given by Eq. (i″), may be written

$$h = r_m \frac{\sin^2 \alpha}{2}, \qquad r = r_m \sin 2\alpha \tag{j'}$$

   Writing the first of Eq. (j′) in the form

$$h = \frac{r_m}{4} (1 - \cos 2\alpha)$$

   We obtain from these equations

$$\cos 2\alpha = 1 - \frac{4h}{r_m}, \qquad \sin 2\alpha = \frac{r}{r_m} \tag{k}$$

   To eliminate the unknown angle α between Eq. (k), we square both sides and add the equations, obtaining

$$1 = \left(1 - \frac{4h}{r_m}\right)^2 + \frac{r^2}{r_m^2}$$

---

[2] For more detailed discussion of the problem, see the authors' "Advanced Dynamics," p. 94, McGraw-Hill, New York, 1948.

in the figure. If q is the weight of the rod per unit length, the intensity of the distributed inertia force is qv^2/gr. For such loading, the maximum bending moment is at the middle of the side rod and has its greatest magnitude when the rod is in the position $A'B'$, for here the inertia load and gravity load are in the same direction perpendicular to the axis of the rod. The magnitude of this moment, calculated as for a simply supported beam, is

$$M_{\max} = \left(q + \frac{qv^2}{gr}\right)\frac{l^2}{8} = \frac{ql^2}{8}\left(l + \frac{v^2}{gr}\right) \tag{a}$$

For the given numerical data $v = 0.375 \times 8\pi$ m/s, and Eq. (a) becomes

$$M_{\max} = \frac{ql^2}{8}(1 + 24.5) \tag{b}$$

This maximum bending moment in the side rod is 25.5 times that due to the weight of the bar alone. Further, from Eq. (a) we see that this moment increases with the square of the speed of operation of the engine. From this example we see that stresses due to inertia load must play an important role in the design of high-speed machinery.

2. Determine the circumferential tension S produced in a uniformly rotating thin circular ring of uniform cross-sectional area A and mean radius r (Fig. 8.14) if the peripheral velocity of the ring is v.

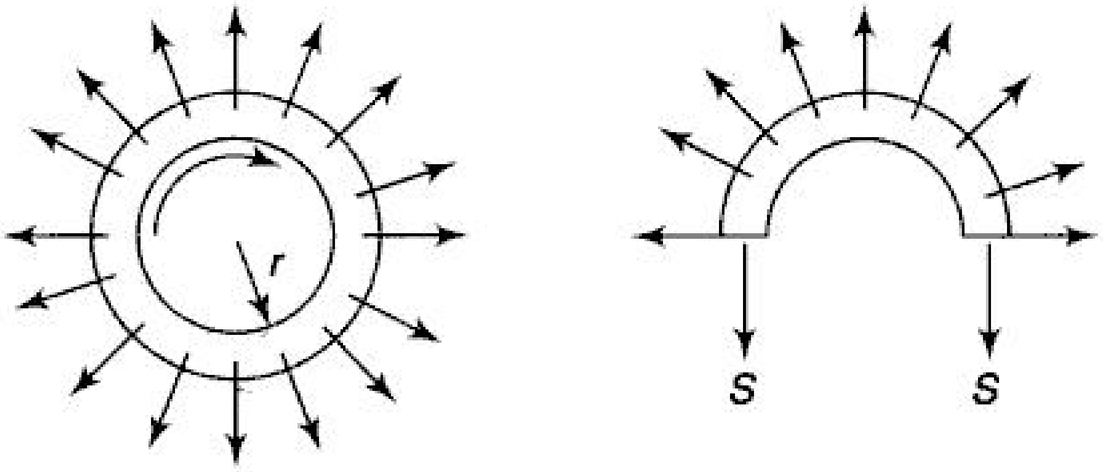

Fig. 8.14

(a) (b)

Solution: If q is the weight per unit length of the ring, the intensity of the uniformly distributed inertia force due to rotation is qv^2/gr and is directed radially outward as shown in the figure. Considering the dynamic equilibrium of one-half the ring as shown in Fig. 8.14(b), we obtain for the circumferential tension (see Section 3.8)

$$S = \frac{qv^2}{gr}r = \frac{qv^2}{g} \tag{c}$$

The corresponding tensile stress is

$$S = \frac{S}{A} = \frac{qv^2}{Ag} = w\frac{v^2}{g} \tag{d}$$

where $\omega = q/A$ is the weight per unit volume of the material of the ring.

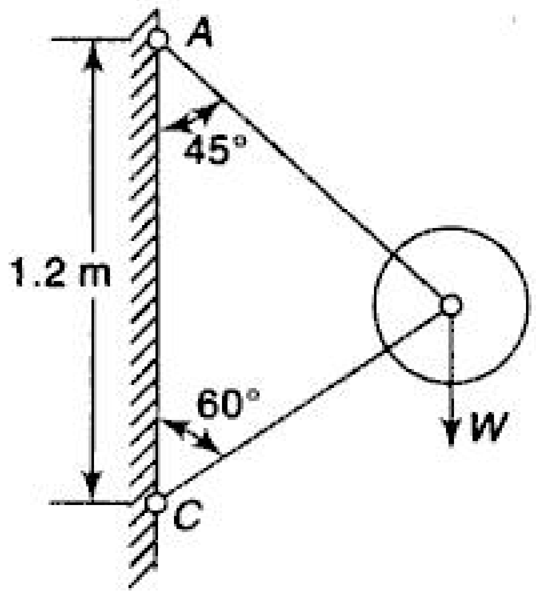
A
45°
1.2 m
60°
W
C

Fig. C

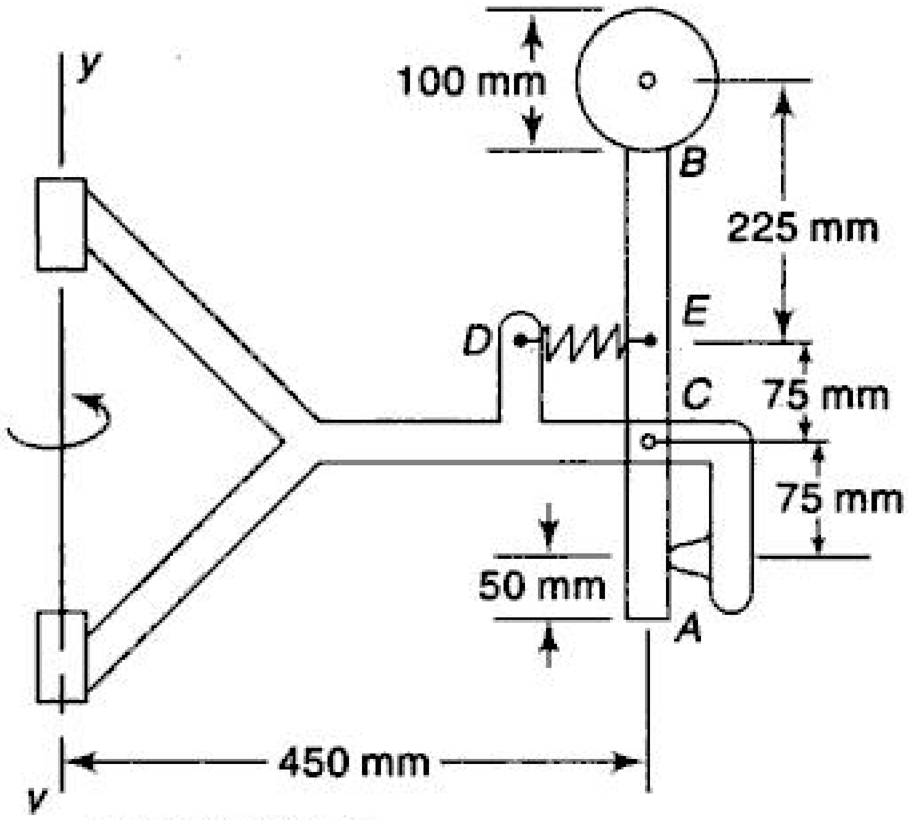
y
100 mm
B
225 mm
D
E
C
75 mm
75 mm
50 mm
A
450 mm
y

Fig. D

the top is 30 N. When the system is at rest, the initial tension in the spring *DE* is 100 N. At what rpm will contact at *A* be broken? Assume the frame and bar *AB* to be absolutely rigid. (*Ans.* 38.7 rpm)

8. At what uniform speed of rotation around the vertical axis *AB* will the balls *C* and *D* of equal weights *W* begin to lift the weight *Q* of the device shown in Fig. E? The following numerical data are given: $W = 44.5$ N, $Q = 89$ N, $l = 250$ mm. Neglect all friction and the weights of the four hinged bars of length *l*. The weight *Q* can slide freely along the shaft *AB*. (*Ans.* $n = 111$ rpm)

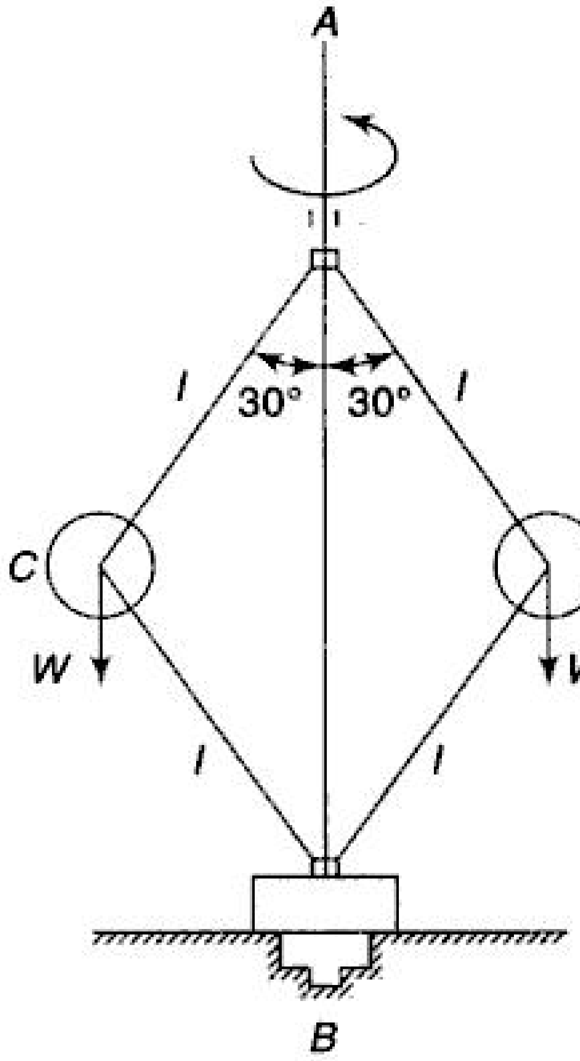
A
l
30°
30°
l
C
D
W
W
l
l
B

Fig. E

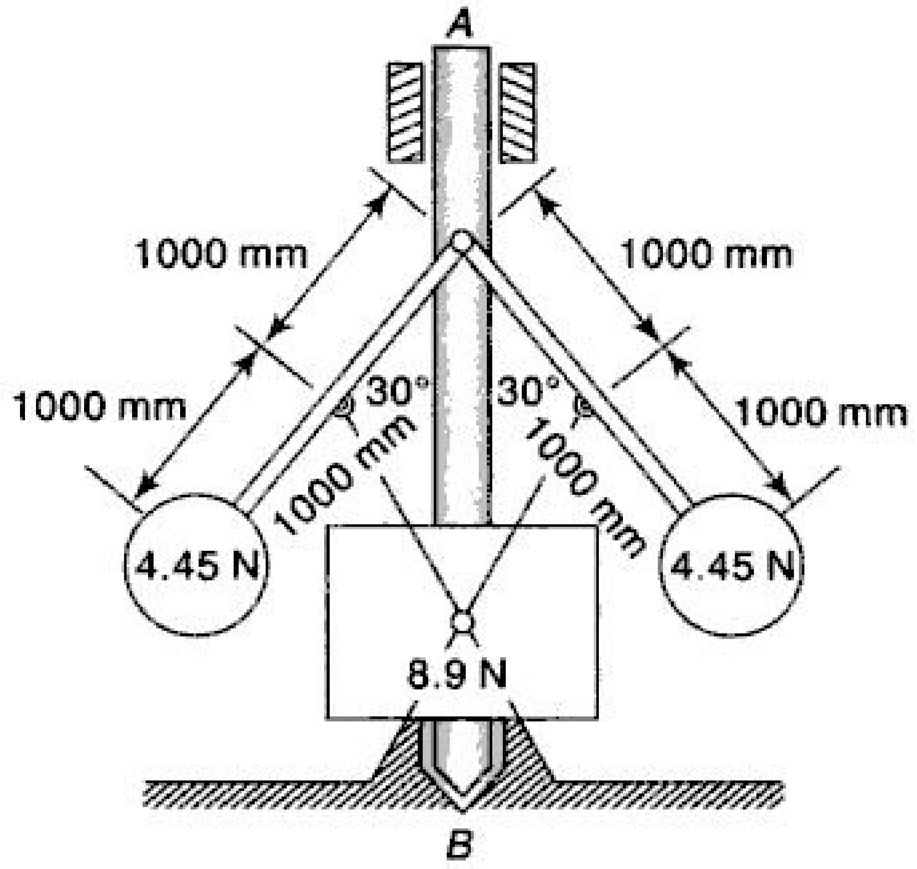
A
1000 mm
1000 mm
1000 mm
1000 mm
30°
30°
1000 mm
1000 mm
4.45 N
4.45 N
8.9 N
B

Fig. F

9. The 8.9 N weight of the governor in Fig. F can slide freely on the vertical shaft *AB*. At what rpm about the axis *AB* will the 4.45 N fly balls lift the sliding weight free of its support? Neglect friction and the weights of the bars. (*Ans.* 101 rpm)

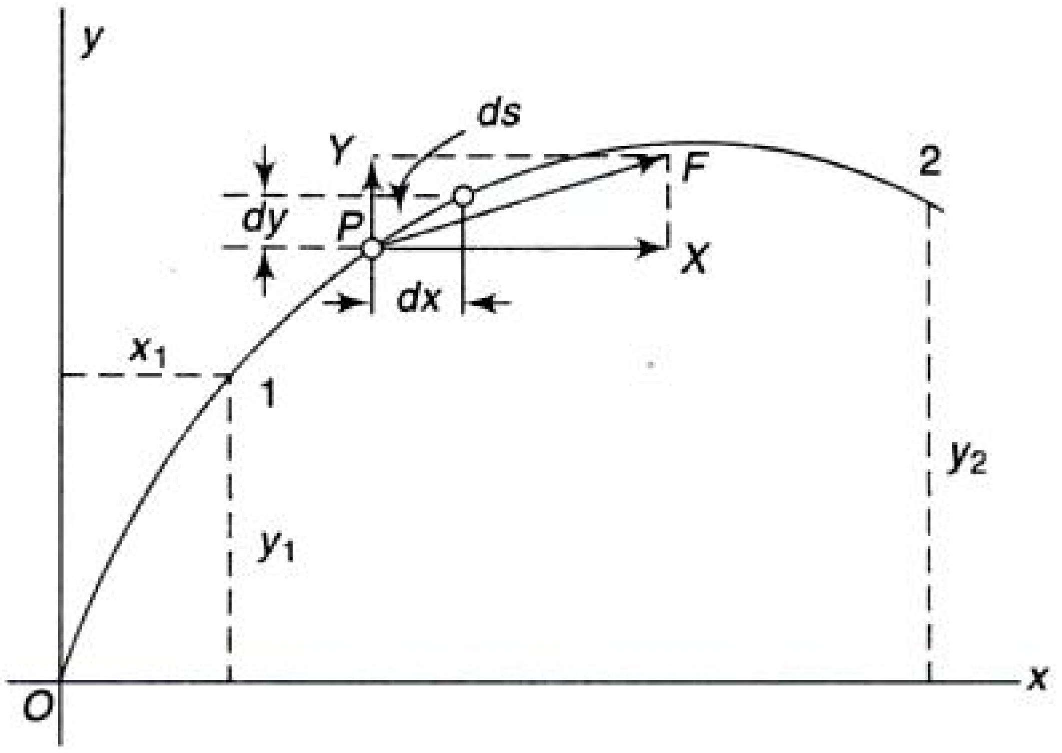

Fig. 8.22

Then since the work of the resultant is equal to the algebraic sum of the works of its components, we conclude that

$$S\,ds = X\,dx + Y\,dy \tag{c}$$

Substituting this in Eq. (b), we obtain

$$d\left(\frac{W}{g}\frac{v^2}{2}\right) = X\,dx + Y\,dy \tag{d}$$

Then integrating as before between limits corresponding to points 1 and 2 on the path, we obtain

$$\frac{Wv_2^2}{2g}\ \frac{Wv_1^2}{2g} = \int_{x_2}^{x_1} Xdx + \int_{y_1}^{y_2} Ydy \tag{61b}$$

We see that Eq. (61b) differs from Eq. (61a) only in the manner of expressing the work done by the acting forces. Usually, the form (61a) will be preferable when the path of the particle is prescribed and the form (61b), when it is not.

The law of conservation of energy as represented by Eq. (49), for an ideal system of particles performing rectilinear motions can be used also in dealing with a system of particles that perform curvilinear motions, provided the conditions of a conservative system are satisfied.

Examples Examples Examples Examples Examples

1. Referring to Fig. 8.23, assume that a particle of weight W starts from rest at B_1 and slides under the influence of gravity along a smooth track B_1B_2 in a vertical plane. Find the velocity of the particle at B_1.

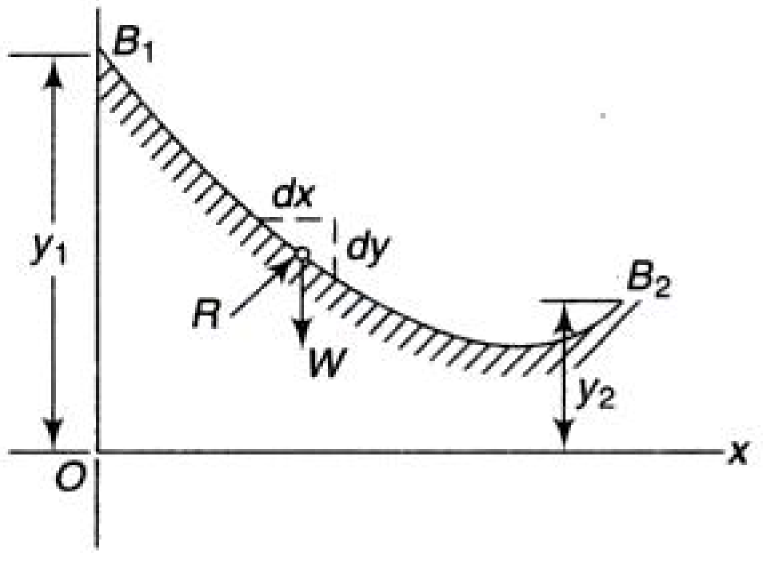

Fig. 8.23

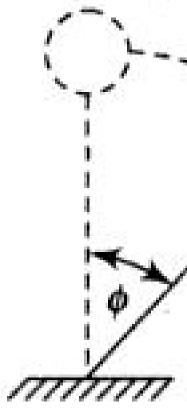

Fig. C

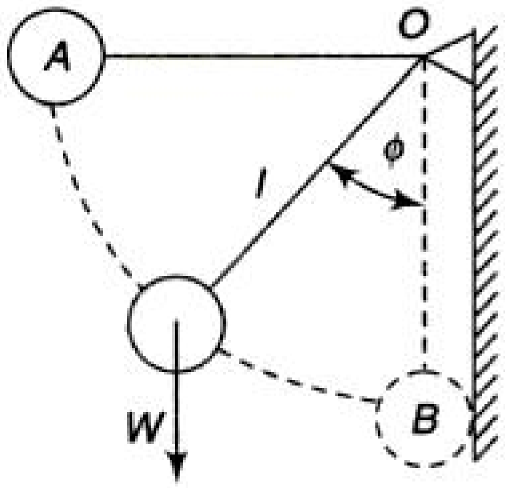

Fig. D

6. A small car of weight W starts from rest at A and rolls without friction along the loop $ACBD$ (Fig. E). What is the least height h above the top of the loop at which the car can start without falling off the track at point B, and for such a starting position what velocity will the car have along the horizontal portion CD of the track? Neglect friction. (*Ans.* $h_{\min} = r/2$; $v_c = \sqrt{5gr}$)

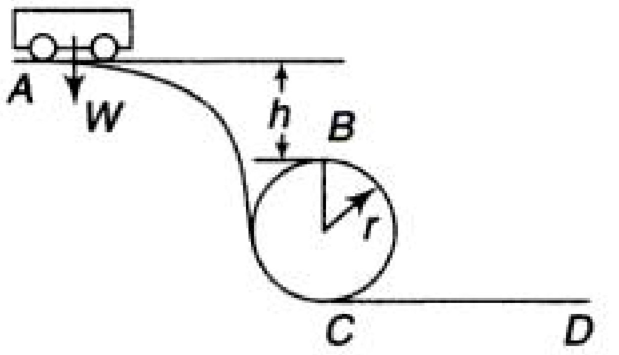

Fig. E

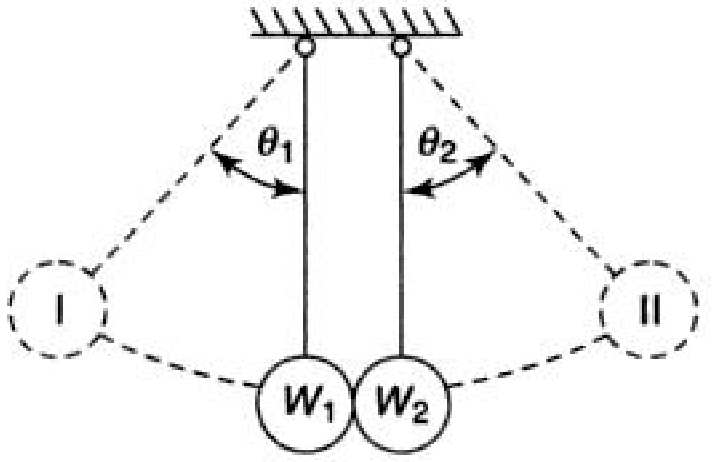

Fig. F

7. Referring to Fig. F, assume that the ball I of weight W is released from rest in the position $\theta_1 = 60°$ and swings downward to where it strikes the ball II of weight $3W$. Assuming an elastic impact, calculate the angle θ_2 through which the larger pendulum will swing after the impact. (*Ans.* $\theta_2 = 28°57'$)
8. In the system shown in Fig. F, the ball I is allowed to swing downward from rest in the position defined by the angle $\theta_1 = 45°$ and to strike the ball II, which after impact, swings upward to the position defined by the angle $\theta_2 = 30°$. If the weights of the balls are equal, find the coefficient of restitution e for the materials. (*Ans.* $e = 0.35$)
9. In Fig. G a small ball of weight $W = 22.25$ N starts from rest at O and rolls down the smooth track OCD under the influence of gravity. Find the reaction R exerted on the ball at C if the curve OCD is defined by the equation $y = h \sin (px/\, l)$ and $h = l/3 = 0.9$ m. (*Ans.* $R = \approx 71$ N)
10. Referring to Fig. H, find the value of the angle ϕ defining the position of the point B where the particle will jump clear of the cylinder surface after the string OA has been cut. Neglect friction. (*Ans.* $\cos \phi = \dfrac{2}{3} \cos \alpha$)

This formula which neglects the cross-sectional dimensions can be used to calculate the period of free swing of a homogeneous slender bar of any prismatic form.

2. The compound pendulum shown in Fig. 9.15 is so constructed that it has the same period τ for small amplitudes of swing about either the knife-edge A [Fig. 9.15(a)] or the knife-edge B [Fig. 9.15(b)]. Prove that the distance l between the knife-edges is the equivalent length of the pendulum if $c \neq l/2$.

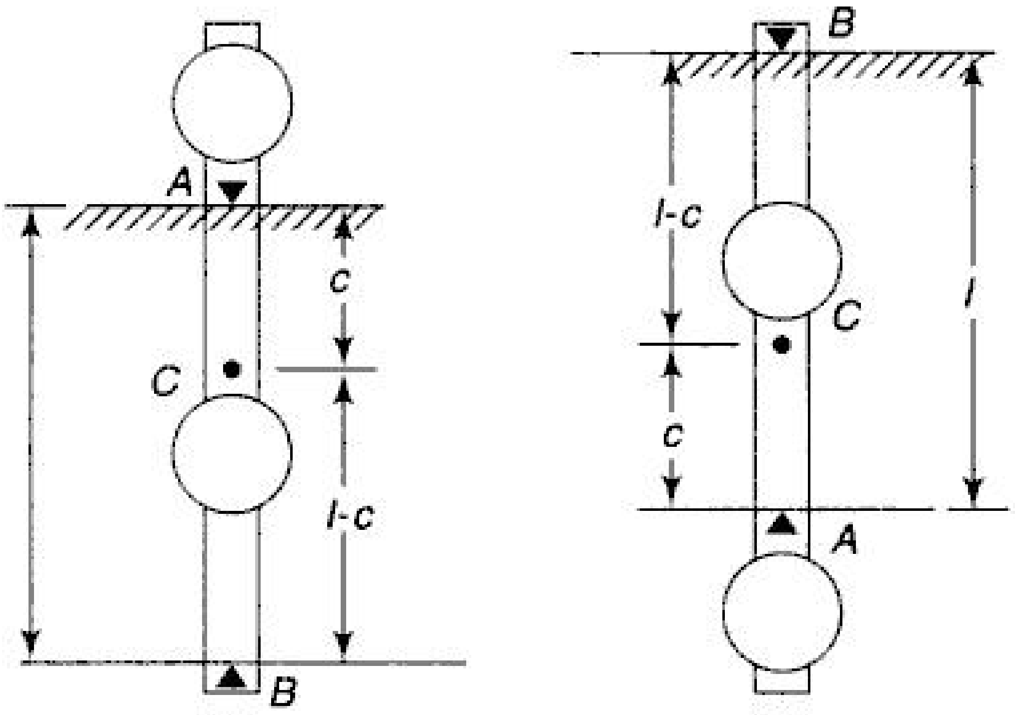

Fig. 9.15 (a) (b)

Solution: For case a, the equivalent length by Eq. (74b) is

$$L_1 = c + \frac{i_c^2}{C} \tag{h}$$

while for case b it is

$$L_2 = (l - c) + \frac{i_c^2}{l - c} \tag{i}$$

And by virtue of the specified equality of periods

$$L_1 = L_2 = L$$

Eliminating i_c^2 between expressions (h) and (i), we obtain

$$(L - c)\,c = (L - l + c)\,(l - c)$$

which may be written in the form

$$l^2 - (L - 2\,c)\,l + 2\,cL = 0 \tag{j}$$

Solving this quadratic for l, obtain

$$l = L \text{ or } l = 2c$$

Since we excluded the possibility of $l = 2c$, we conclude that l is the equivalent length of the pendulum.

3. When hanging from a nail at A, the T square in Fig. 9.16 has an observed period $\tau = 1.5$ s. A simple static balance test shows that the center of gravity C is at the distance $a = 600$ mm the end A of the rule. At what distance x from A should a hole O be drilled in the rule so that the period of free swing about this point of suspension will be the minimum possible period?

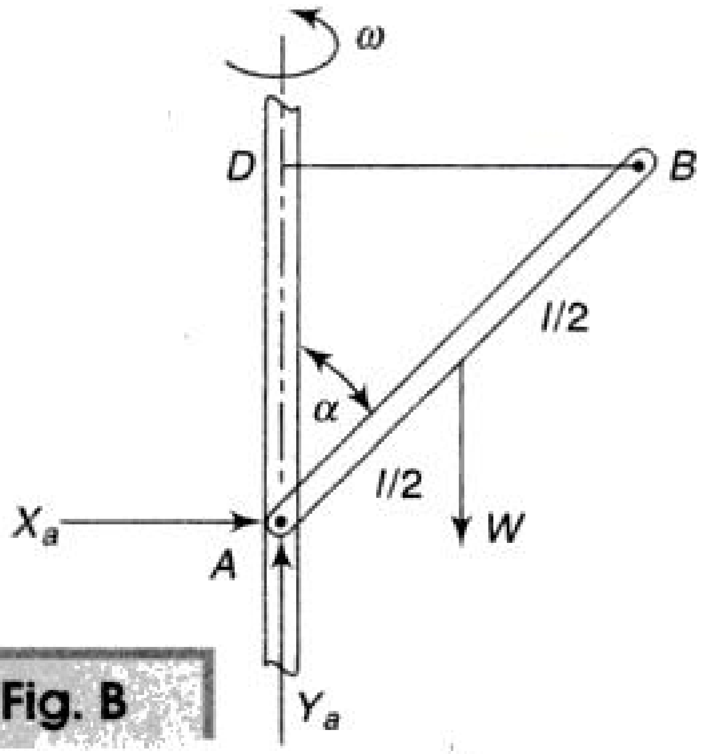

Fig. B

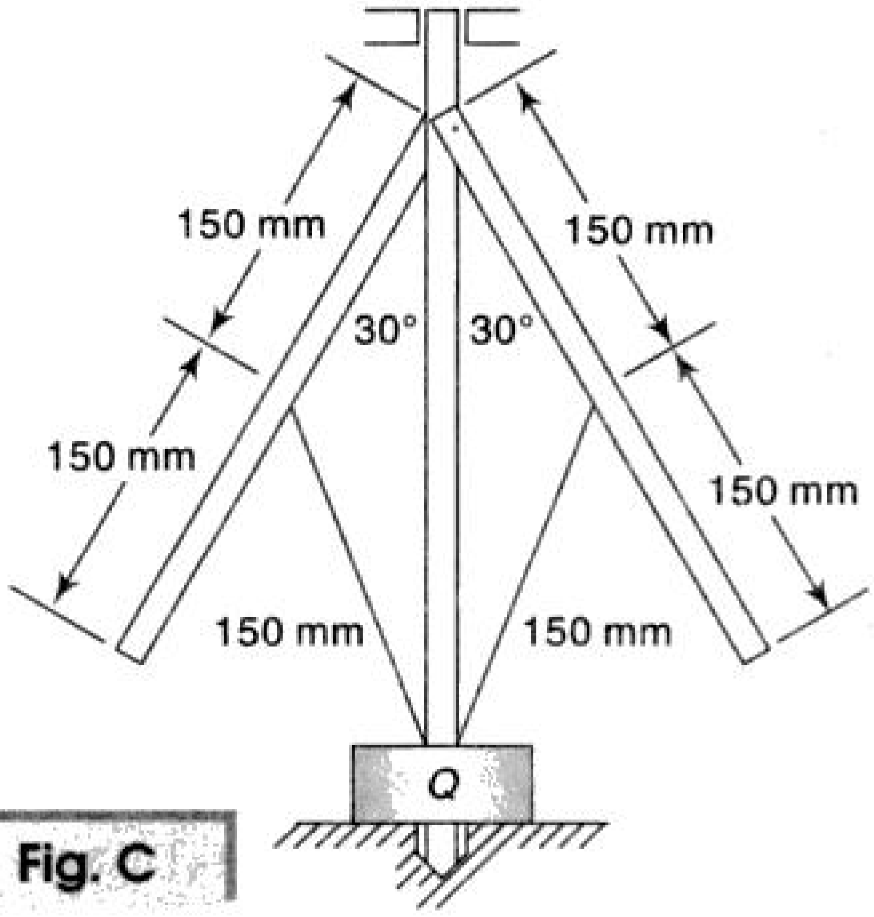

Fig. C

7. Referring to Fig. C, calculate the angular speed at which the sliding weight $Q = 8.9$ N will begin to lift free of its support if each of the slender prismatic bars weighs 8.9 N. (*Ans.* 101 rpm)
8. A thin circular disk *CD* of radius r and weight W is attached at its center to a shaft *AB*, and its plane makes with the plane normal to the axis of the shaft a small angle α (Fig. D). If the disk rotates with constant angular velocity ω, find the bearing reactions at A and B due to this rotation.
   Hint. Use Eq. (d) from Example 4 above for an elemental ring of radius ρ and thickness $d\rho$ and integrate over the full radius r of the disk.
   (*Ans.* $R_a = R_b = W\omega^2 r^2 a/4gl$)

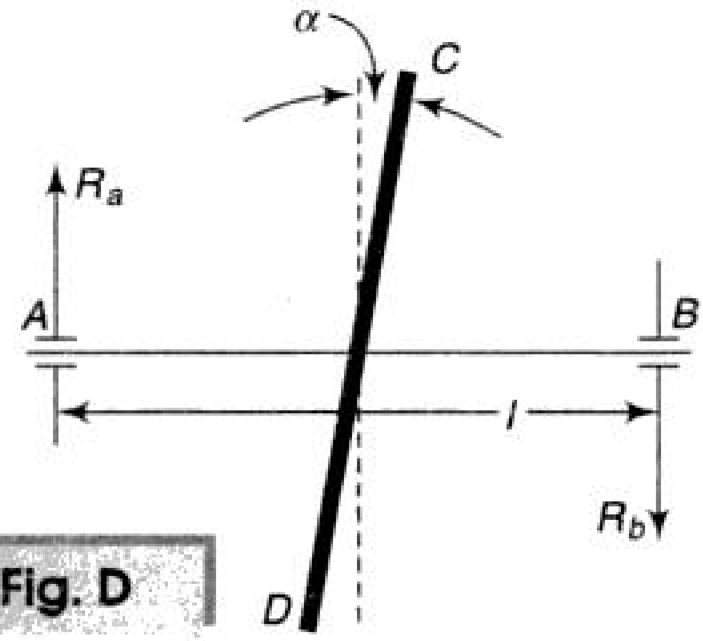

Fig. D

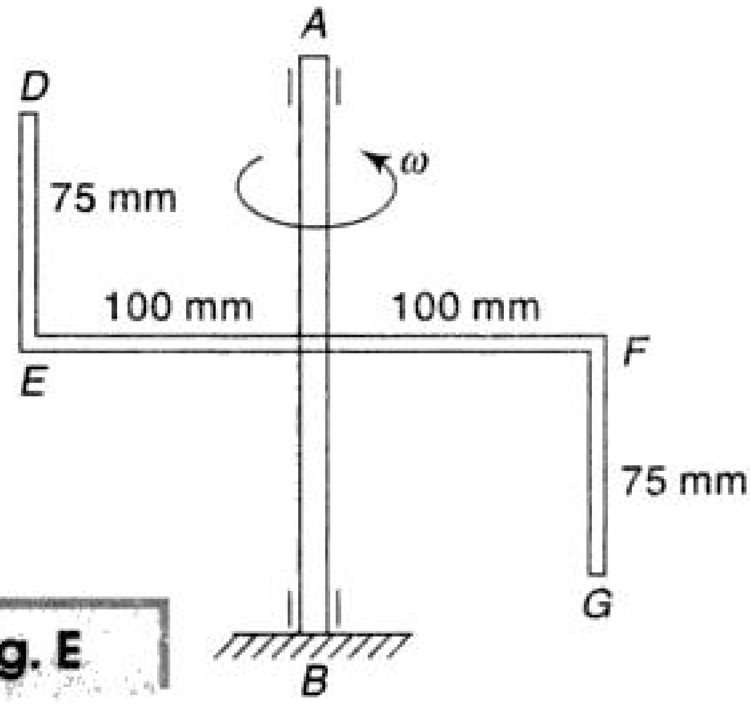

Fig. E

9. Referring to Fig. E, calculate the bending couple M transmitted to the vertical shaft *AB* by the bent prismatic rod *DEFG*. The weight per unit length of the rod is $w = 17.8$ N/m and the system rotates at a constant angular speed of 600 rpm. (*Ans.* $M = 4.1$ N m)
10. An ordinary carpenters square (450 by 600 mm) hangs freely from a pin at the end of the long leg and rotates with constant angular velocity about a vertical axis through this pin. The weight of the short leg is 1.113 N; that of the long leg, 3.34 N. What is the required rpm to keep the long leg vertical, i.e. on the axis of rotation? (*Ans.* 38.2 rpm)

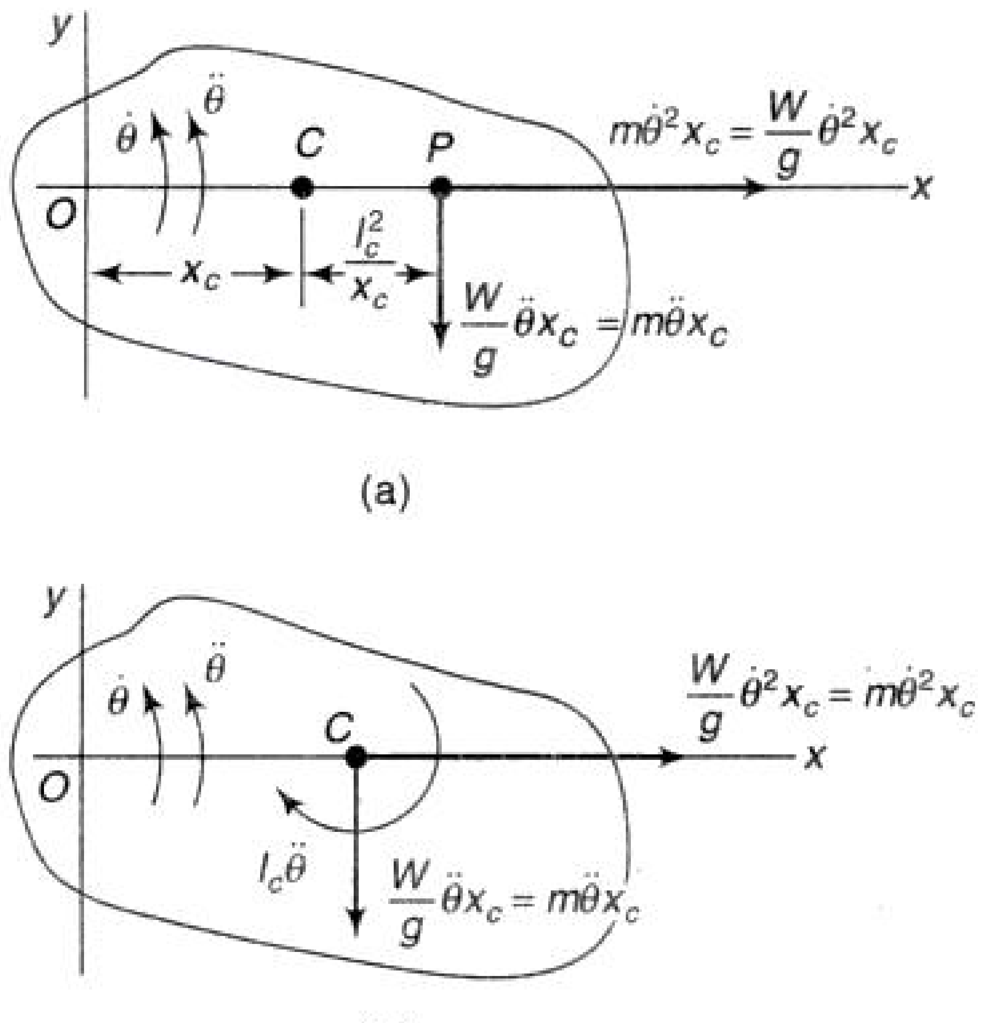
y
$\dot{\theta}$
$\ddot{\theta}$
C
P
O
$m\dot{\theta}^2 x_c = \frac{W}{g}\dot{\theta}^2 x_c$
x
x_c
$\frac{I_c^2}{x_c}$
$\frac{W}{g}\ddot{\theta}x_c = m\ddot{\theta}x_c$
(a)
y
$\dot{\theta}$
$\ddot{\theta}$
C
O
$\frac{W}{g}\dot{\theta}^2 x_c = m\dot{\theta}^2 x_c$
x
$I_c\ddot{\theta}$
$\frac{W}{g}\ddot{\theta}x_c = m\ddot{\theta}x_c$
(b)

Fig. 9.28

Examples Examples Examples Examples Examples

1. A homogeneous thin plate in the form of a circular quadrant of radius r is supported in a vertical plane by a hinge at O and a stop at B as shown in Fig. 9.29(a). Find the horizontal and vertical components of the reaction at the hinge O: (a) before the stop at B is removed and (b) just after it is removed.

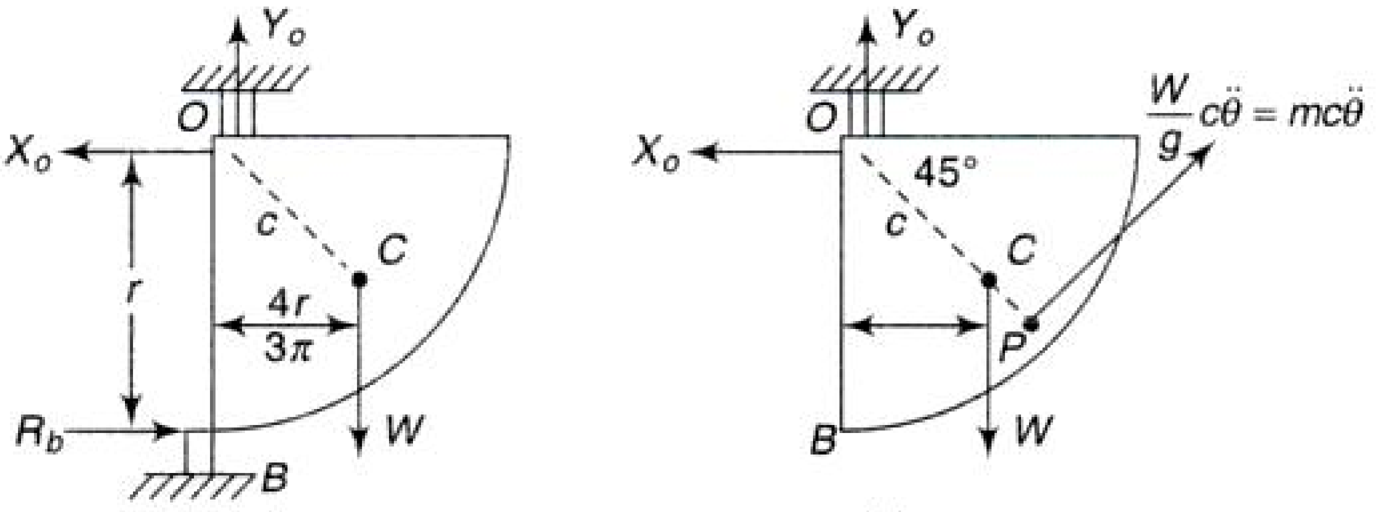
Y_o
O
X_o
c
C
r
$\frac{4r}{3\pi}$
R_b
W
B
(a)
Y_o
O
X_o
45°
c
C
P
$\frac{W}{g}c\ddot{\theta} = mc\ddot{\theta}$
B
W
(b)

Fig. 9.29

Solution: Before the stop at B is removed, the plate is in static equilibrium, and we have the free-body diagram as shown in Fig. 9.29(a). Writing $\Sigma(M_b)_i = 0$ and $\Sigma Y_i = 0$, we obtain

$$X_0 = \frac{4W}{3\pi} = 0.424W, \qquad Y_0 = W \tag{d}$$

directed as shown.

10. A slender weightless rod pivoted at C has balls of weights $W_1 = 22.25$ N and $W_2 = 13.35$ N attached at its ends, as shown in Fig. H, and is initially at rest in the unstable position shown in the figure. If the system is disturbed slightly, the bar begins to rotate about C. Find the horizontal and vertical components of the reaction at C as the bar passes through the horizontal position. (*Ans.* $X_c = -11.13$ N; $Y_c = +4.45$ N)

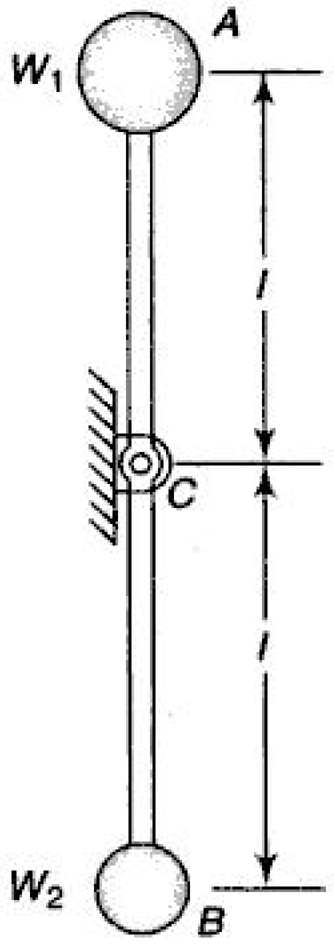

Fig. H

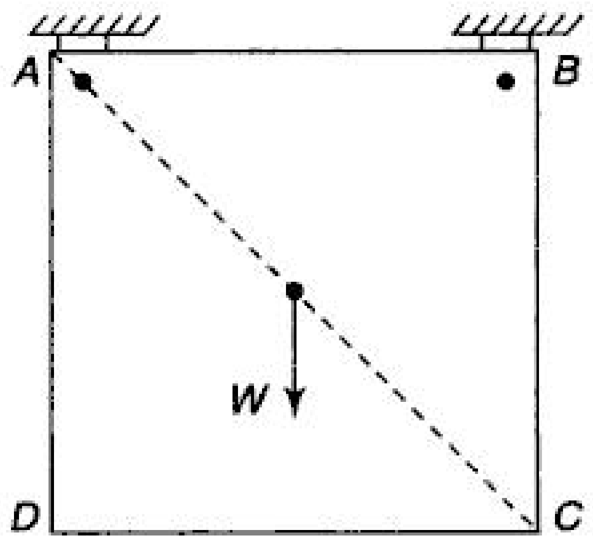

Fig. I

11. A homogeneous plate 0.3 m square is supported in a vertical plane as shown in Fig. I. If the pin at B is removed, what angular velocity ω will the plate acquire by the time the diagonal AC becomes vertical? (*Ans.* $\omega = 4.47$ rad/s)
12. Referring again to Fig. I, calculate the vertical reaction R_a for the instant when the diagonal AC of the square is vertical. (*Ans.* $R_a = 1.44\ W$)
13. The ballistic pendulum shown in Fig. 9.35, Example 3, Section 9.9 has a total weight $W = 89$ N, and the distance from its center of gravity to the center of suspension is found by experiment to be $c = 2.289$ m. Also its observed period of oscillation for small amplitudes is $\tau = 3.22$ s. When the pendulum is hanging in its position of stable equilibrium, a rifle bullet of weight ($w = 0.278$ N) is fired horizontally into the block at the center of oscillation, and as a result of the impact the pendulum is observed to swing through an angle $\theta_m = 26°40'$ before coming to rest. Find the muzzle velocity v of the bullet. (*Ans.* $v = 660$ m/s)

9.11 GYROSCOPES

A *gyroscope* usually has the form of a solid of revolution mounted on an axle coinciding with its geometric axis. If such a body rotates about this axis of symmetry (*axis of spin*) with a high angular velocity, it possesses certain dynamic characteristics that are of practical interest. In discussing these characteristics, it is advantageous to use the principle of angular momentum and the idea of vectorial representation of resultant angular momentum with respect to a fixed point and resultant moment of external forces with respect to the same point as discussed in

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Examples Examples Examples Examples Examples

1. A right circular cylinder rolls without slipping along a horizontal plane AB, and its center has at a certain instant a velocity v_c, as shown in Fig. 10.2(a). Find the velocities at the same instant of the points D and E on the rim of the cylinder.

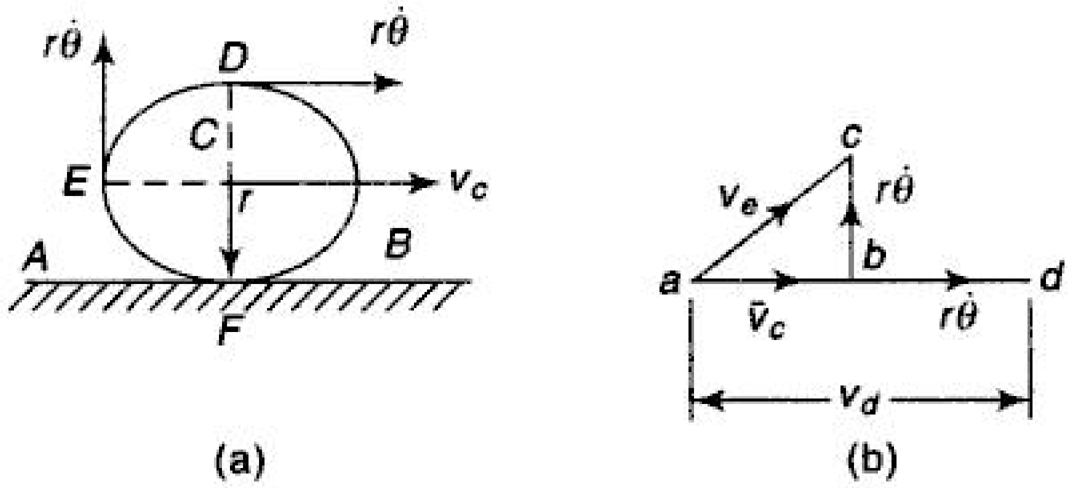

Fig. 10.2 (a) (b)

Solution: Since the cylinder rolls without slipping, the velocity of point F in contact with the plane must be zero. Then using Eq. (84) with point C as a pole, we have $v_f = v_c - r\dot{\theta} = 0$ from which $\dot{\theta} = \frac{v_c}{r}$. Then owing to rotation, the points D and E have, with respect to the moving pole C, relative velocities $r\dot{\theta} = rv_c/r = v_c$ directed as shown in Fig. 10.2(a). Adding geometrically each of these relative velocities to the velocity v_c of the pole C, we obtain the vectors $\overline{ad}$ and $\overline{ac}$ as shown in Fig. 10.2(b) and representing, respectively, the velocities of points D and E. From Fig. 10.2(b), we see that the velocity of point D has the magnitude $2v_c$ and the velocity of point E, the magnitude $\sqrt{2}v_c$.

2. A prismatic bar AB has its ends A and B constrained to move horizontally and vertically as shown in Fig. 10.3(a). If the end A of the bar moves with constant velocity v_a, find the angular velocity $\dot{\theta}$ of the bar and the velocity v_b of the end B for the instant when the axis of the bar makes the angle θ with the horizontal x-axis.

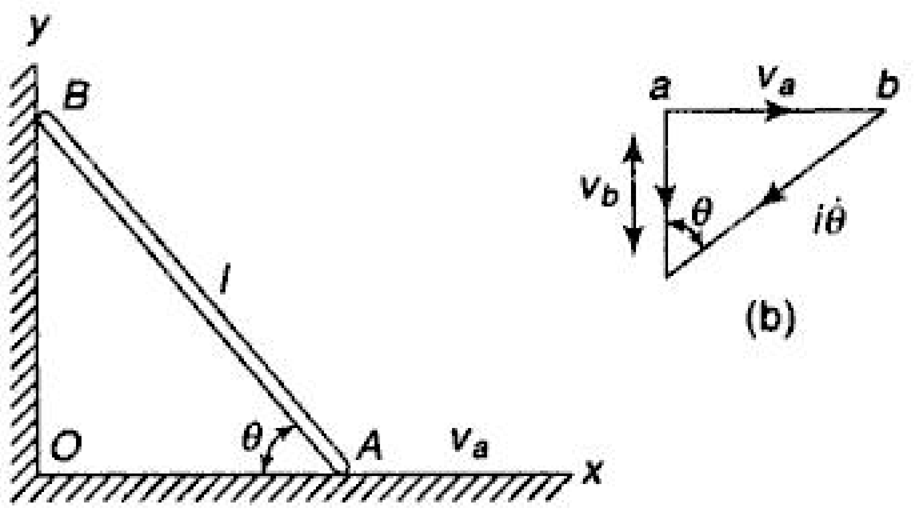

Fig. 10.3 (a) (b)

there has been no appreciable change in the position of the body. In dealing with such problems, it is customary to neglect entirely all ordinary forces such as gravity in comparison with the very large impact forces and also to neglect change in position of the body during the impact. In short, knowing the motion of a body just before impact and the nature of the impact, what will be the new motion of the body just after impact?

As an example, let us consider a solid right circular cylinder that rolls with constant velocity along a horizontal plane and suddenly strikes an obstruction at B as shown in Fig. 10.18.

Just before impact at B, the cylinder has a horizontal velocity v_c and angular velocity $\dot{\theta} = v_c/r$, since point A is its instantaneous center of rotation. Just after the impact, point B becomes the new instantaneous center, and we wish to know the magnitude of the new velocity v_c' of the cylinder. To find this, we observe that during the interval of impact very large forces X_b and Y_b act on the cylinder at point B. In comparison with such forces, we can neglect the gravity force W entirely and assume that X_b and Y_b are the only forces acting during the impact. Then if we take B as a moment center, the moments of these forces vanish and we conclude from Eq. (89) that the angular momentum of the cylinder with respect to point B does not change. Before impact it was

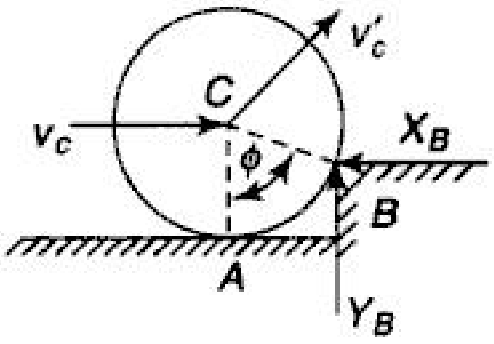

Fig. 10.18

$$H_b = \frac{W}{g}\frac{r^2}{2}\frac{v_c}{r} + \frac{W}{g} v_c r \cos\phi \tag{g}$$

while just after impact it is

$$H_b' = \frac{W}{g}\frac{r^2}{2}\frac{v_c'}{r} + \frac{W}{g} v_c' r \tag{h}$$

Equating expressions (g) and (h), we obtain

$$v_c' = \frac{v_c}{3}(1 + 2\cos\phi) \tag{i}$$

The above discussion assumes that there is sufficient friction at B to prevent slipping during the change in motion.

Examples Examples Examples Examples Examples

1. A thin homogeneous plate of any shape has a prescribed motion in its own plane defined by the velocity components $\dot{x}_c, \dot{y}_c$ of its center of gravity C and its angular velocity ω (Fig. 10.19). If a certain point O in the plate at the distance r from C is suddenly fixed by means of a pin, find the new angular velocity of the plate around this point.

   Solution: Through point O, we choose fixed coordinate axes x, y, in the plane of motion such that the x-axis coincides with the instantaneous posi-

*10. A thin steel hoop of weight W and radius r starts from rest at A and rolls down along a circular cylindrical surface of radius a as shown in Fig. G. Determine the angle φ defining the position of point B where the hoop will begin to slip if the coefficient of friction at the point of contact is $\mu = \frac{1}{3}$. (*Ans.* $\varphi = 29°32'$)

*11. Referring to Fig. G, assume that the roller is a gear wheel with pitch radius r and radius of gyration i_c rolling on a cylindrical rack with patch radius a so that there is no possibility of slipping even without friction. Under these conditions, find the value of the angle j at which the roller will jump clear of the ract if it starts from rest at A. Data are given as follows: $a = 300$ mm; $r = 100$ mm ; $i_c = 75$ mm. (*Ans.* $j = 55°51$)

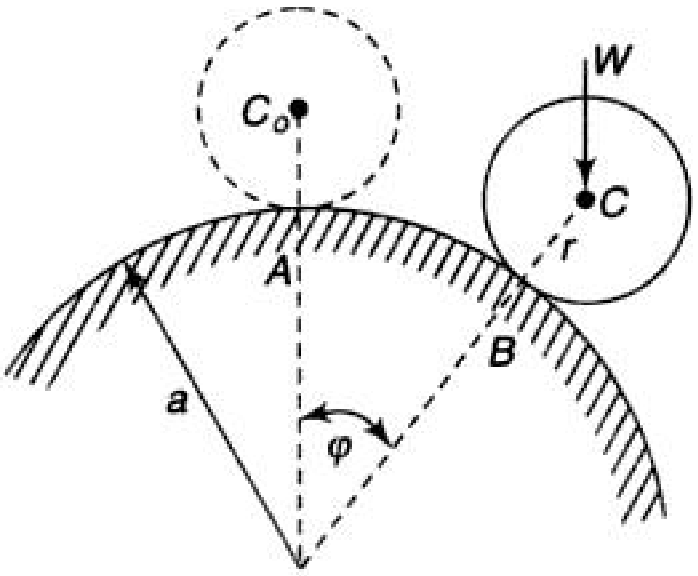

Fig. G

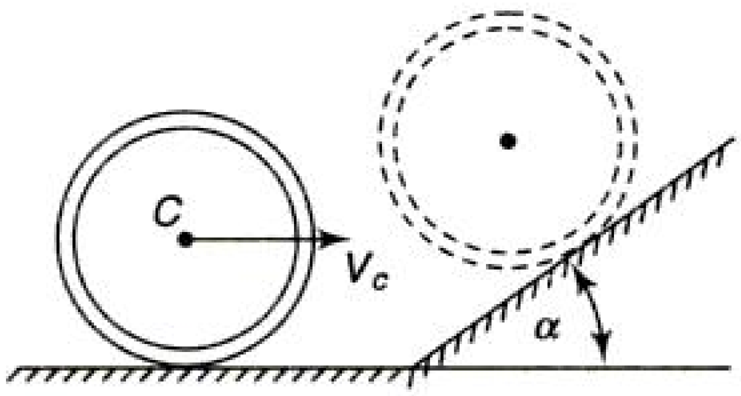

Fig. H

12. The thin cylindrical shell in Fig. H approaches the inclined plane at the right with velocity v_c as shown, rolls up this plane, until brought to rest by gravity, and then rolls back down again. What will be its velocity v_c'' as it finally rolls to the left along the horizontal plane? Assume that the shell at all times rolls without slipping and that there is no rebound after impacts. $\left(Ans.\ v_c'' = \frac{1}{4} v_c\,(2 - \cos\alpha)^2\right)$

Solution: The relative acceleration of the particle is directed toward the center of the disk and is

$$a_r = \frac{v_r^2}{r} \tag{f}$$

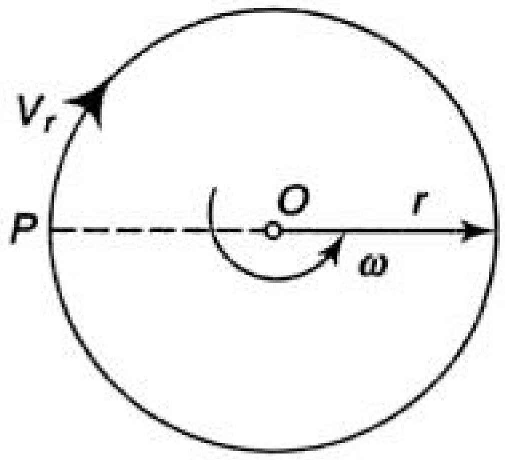

Fig. 11.3

The acceleration of a point on the disk coinciding with an instantaneous position of the particle is also directed toward the center of the disk, and this base acceleration is

$$a_b = \omega^2 r \tag{g}$$

Using the graphical conception of supplementary acceleration as discussed in connection with Fig. 11.2, we conclude that in this case the supplementary acceleration is

$$a_s = 2\omega v_r \tag{h}$$

and that it has the radial direction away from the center. Then the absolute acceleration, from Eqs (f), (g) and (h), is

$$a_r = \frac{v_r^2}{r} + \omega^2 r - 2\omega v_r = r\left(\frac{v_r}{r} - \omega\right)^2 \tag{i}$$

directed toward the center of the disk. When $v_r = r\omega$, the particle is immovable with respect to fixed axes, and in such case, as is seen from Eq. (i), its absolute acceleration is zero.

2. A particle P moves with uniform relative velocity v_r along a meridian of a sphere of radius r which rotates about a fixed vertical diameter with uniform angular velocity ω (Fig. 11.4). Find its absolute acceleration for the position defined by the angle φ.

Solution: The point of the sphere with which the instantaneous position of the particle P coincides is evidently rotating in a circular path of radius (r cos φ) and with uniform angular velocity ω. Hence the base acceleration is

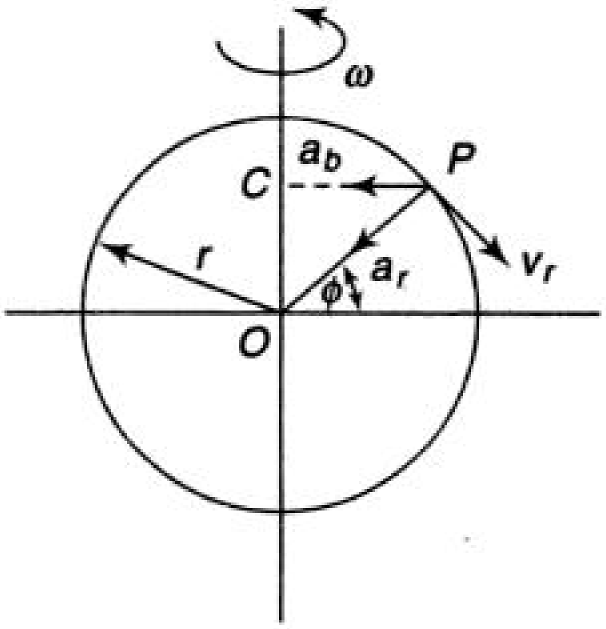

Fig. 11.4

$$a_b = \Omega^2 r \cos\varphi \tag{j}$$

directed as shown in the Fig. 11.4.

Since the relative motion of the particle is uniform along the meridian, the acceleration due to relative motion is directed toward the center of the sphere as shown in the figure and has the magnitude

$$a_r = \frac{v_r^2}{r} \tag{k}$$

The supplementary acceleration of the particle, equal to the doubled velocity of the end of the vector v_r rotating with uniform angular velocity Ω

the relative velocity v_r and relative acceleration a_r vanish. Then in Eq. (93) only the base acceleration a_b remains and the equation reduces to

$$\bar{F} - m\bar{a}_b = 0 \tag{94}$$

Thus we conclude that for *relative equilibrium*, not the resultant force F, but the resultant force F together with the inertia force due to the base acceleration must be zero. Equation (94) like Eq. (93) is expressed in vectorial form. We conclude from it that in the case of relative equilibrium of a particle with respect to a moving body, the algebraic sum of the projections, on any axis, of all real forces acting on the particle together with the inertia force due to acceleration a_b must be zero.

As an example of the formulation of the equations of relative motion for a particular case, let us consider motion of the system shown in Fig. 11.5.

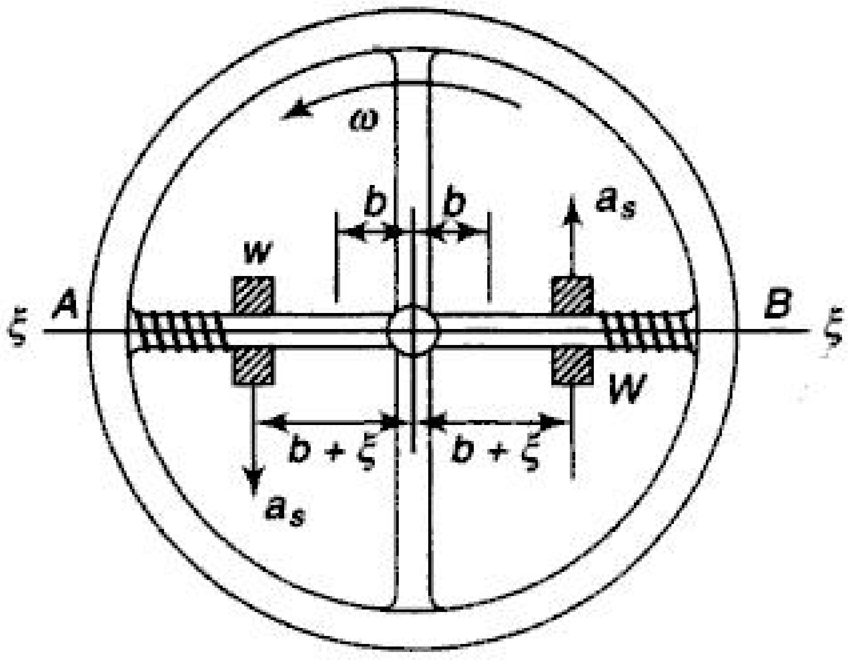

Fig. 11.5

This system consists of two equal weights W that can slide without friction along two spokes of a flywheel uniformly rotating in a horizontal plane with angular velocity ω as shown. Each weight is attached to the rim of the wheel by a helical spring having a spring constant k.

In investigating possible relative motion of the two weights W, we denote by b the distance of the center of gravity of each weight from the axis of rotation of the flywheel when the springs are unstrained. We also assume that the weights are always equidistant from the axis of rotation. Then their positions along the spokes at any instant are completely defined by the relative displacement $b + \xi$ from the center of rotation, ξ being considered positive in the direction away from the axis. The relative acceleration of each weight then is $\ddot{\xi}$. The base acceleration a_b is directed toward the center of the wheel and has the magnitude $\omega^2(b + \xi)$. The supplementary acceleration a_s is directed perpendicular to the diameter AB and is equal to $2\omega\dot{\xi}$. The direction of this acceleration for each weight is shown in the figure, assuming that $\dot{\xi}$ is positive, that is, that the weights are moving toward the rim of the wheel.

Using Eq. (93) and projecting all forces and accelerations onto the diameter AB, we find for the equation of relative motion of either weight

$$\frac{W}{g}\omega\dot{\xi} = -k\xi + \frac{W}{g}(b + \xi)\omega^2 \tag{a}$$

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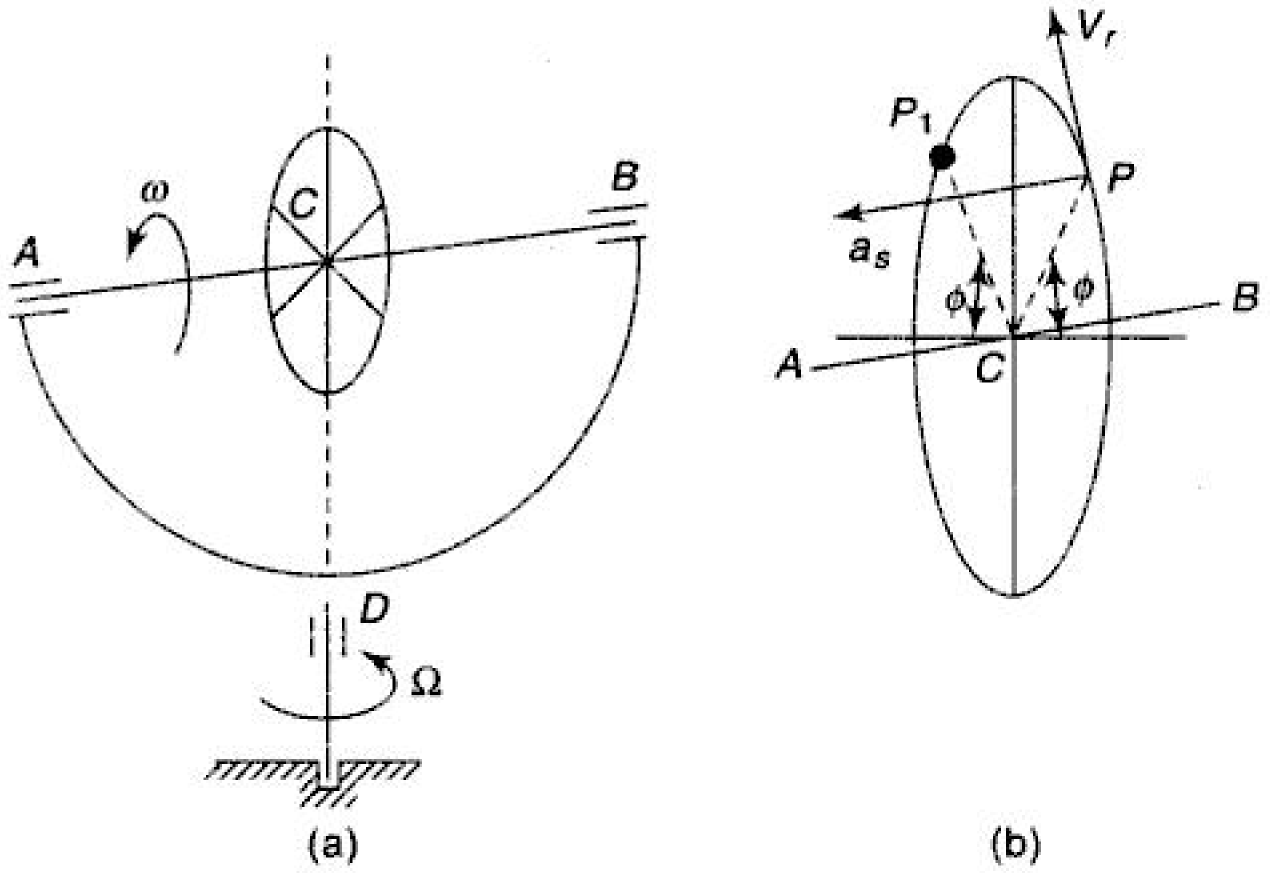

(a) (b)

Fig. 11.10

velocity Ω around a vertical axis. We shall assume also that the entire mass of the wheel is concentrated in its rim of radius r.

Considering a particle P of the rim [Fig. 11.10(b)] and noting that the relative motion of the wheel is rotation with uniform angular velocity Ω about the axis AB, we conclude that the relative velocity of P is $v_r = r\omega$ directed as shown in the figure. Hence the relative acceleration of P is $a_r = \omega^2 r$, directed toward the center C of the wheel. The corresponding inertia forces for all such particles as P are symmetrically distributed around the rim and form a system of forces in equilibrium. Thus they may be disregarded.

In calculating the base acceleration a_b for the particle P, we consider only its motion due to rotation of the frame around the vertical axis. Denoting by φ the angle between the radius CP and the horizontal diameter of the wheel [Fig. 11.10(b)], we see that, owing to rotation of the frame around the vertical axis, the point describes a horizontal circle of radius $r \cos \varphi$. Hence the base acceleration of P is $a_b = \Omega^2 r \cos \varphi$. The corresponding inertia force of the particle P will be in equilibrium with that of a symmetrically situated particle P_1 [Fig. 11.10(b)]. Thus for all particles of the rim, the inertia forces due to base acceleration represent again a system of forces in equilibrium and can be disregarded.

There remain to be considered inertia forces due to supplementary accelerations of the various particles of the rim. From the direction and magnitude of the relative velocity v_r of the particle P, we conclude that for this particle the acceleration a_s is parallel to the axis AB, equal to $2\tau\omega\Omega \sin \omega$ and directed as shown in Fig. 11.10(b). The corresponding inertia force is acting in the opposite direction and gives a moment with respect to the horizontal diameter of the wheel equal to

$$2\, dm\, r\omega\Omega \sin \varphi\, r \sin \varphi = 2\, dm\, \omega\Omega r^2 \sin^2\varphi \qquad \text{(a)}$$

Sometimes I_x and I_y are called the second moment of the area about the axes because its distance is squared from the corresponding axis. The dimension for moment of inertia of area is length to the fourth power. Therefore, SI unit of moment of inertia of area with respect to an axis in its plane is mm^4 or m^4. The first moment of an area can be positive, negative or zero, but its moment of inertia is always positive since both x and y in Eq. (1) are squared. Another point to be noted here is, the first moment of an area is equal to the area times the centroidal distance, but the second moment of an area is not equal to the area times the centroidal distance squared since the square of the mean is less than the mean of the squares.

In simple cases, the integrals (1) can readily be calculated analytically. For the calculation of moment of inertia, the choice of coordinates to use is important. Rectangular coordinates are used for shapes whose boundaries are most easily expressed in these coordinates. Polar coordinates will simplify problems involving boundaries that are easily described in r and θ. The choice of an infinitesimal element of area which simplifies the integration as much as possible is also important. The details of the integration for plane areas in Eq. (1) depend on the choice of the area element dA. The area element dA is a thin strip chosen parallel to the x-axis to compute I_x, so that all of the points of the strip are at the same distance y from the x-axis [Fig. AI.2(a)]. The moment of inertia dI_x of the strip is then obtained by multiplying the area dA of strip by y^2.

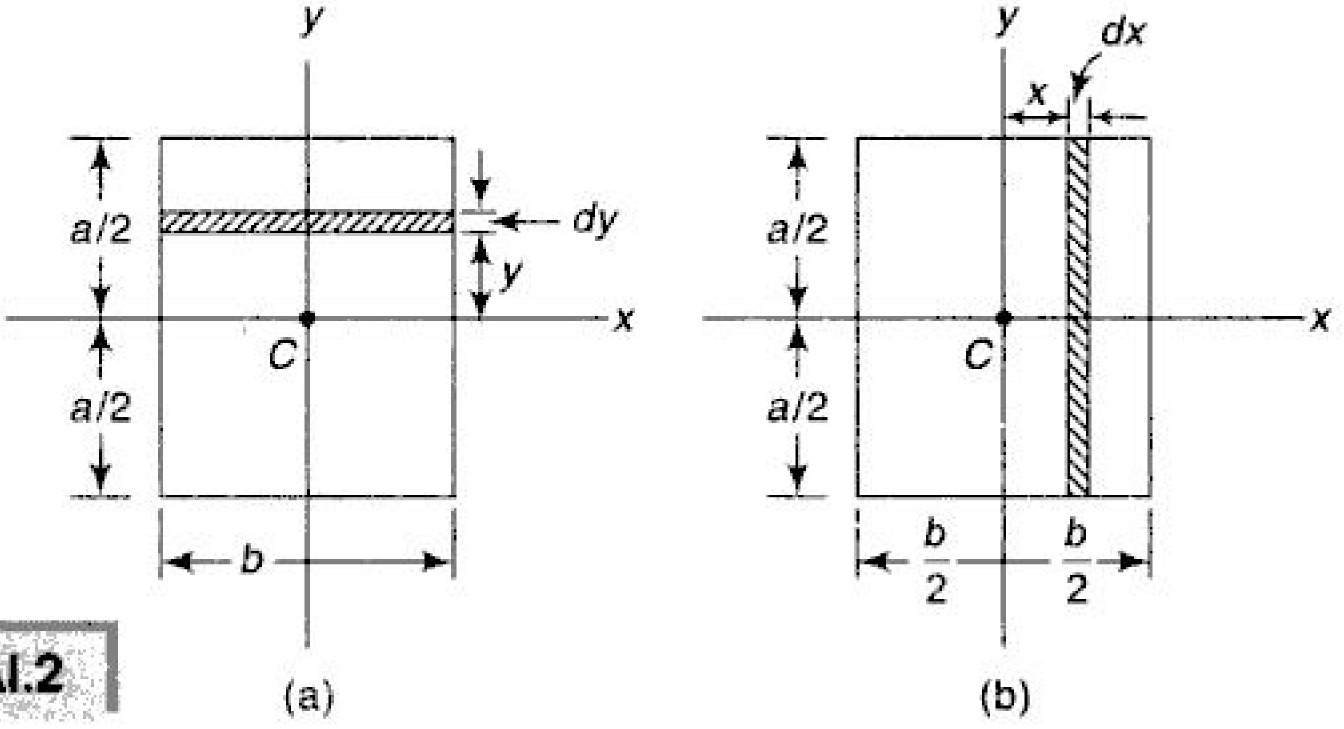

Fig. AI.2

Similarly, the area element dA is a thin strip chosen parallel to the y-axis to compute I_y so that all of the points of strip are at the same distance x from the y axis [Fig. AI.2(b)]. The moment of inertia dI_y of the strip is then obtained by multiplying the area dA of strip by x^2. Mathematically, by integrating, we will get

$$I_x = \int y^2\,dA, \qquad I_y = \int x^2\,dA, \tag{b}$$

Take, for example, the case of a rectangle as shown in Fig. AI.3. In calculating the moment of inertia of this figure with respect to the horizontal axis of symmetry, which is taken as the x-axis, we divide the area of the rectangle into infinitesimal elements like the shaded strip shown in the figure. Then $dA = b\,dy$, and we obtain

$$I_x = 2\int_0^{-a/2} by^2\,dy = \frac{ba^3}{12} \tag{c}$$

3. Find the polar moment of inertia of the shaded area shown in Fig. AI.10 with respect to point O. (*Ans.* $J_0 = 0.274r^4$)
4. Find the polar moment of inertia of the area of the rectangle shown in Fig. AI.3 with respect to one corner. (*Ans.* $J_0 = (a^2 + b^2)ab/3$)
5. Find the polar moment of inertia of the area of a circular sector of radius r and central angle α with respect to its center. (*Ans.* $J_0 = \alpha r^4/4$)

A.3 PARALLEL-AXIS THEOREM

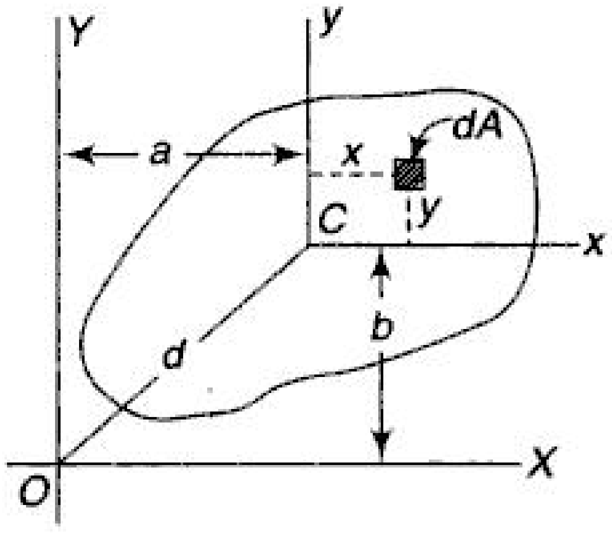

Fig. AI.12

In Fig. AI.12, let X, Y be rectangular coordinate axes through any point O in the plane of the figure and x, y correspondingly parallel axes through the centroid C of an area as shown.

Then by definition, the moment of inertia of the area with respect to the X axis is

$$I_x = \int (y + b)^2 dA = \int y^2 dA + 2b \int y dA + b^2 \int dA \tag{r}$$

Noting that $\int y dA = 0$, since the x-axis passes through the centroid C of the area, Eq. (r) reduces to

$$I_X = \bar{I}_x + Ab^2 \tag{5a}$$

where the notation $\bar{I}_x = \int y^2 dA$ is used to denote the centroidal moment of inertia of the area with respect to the x-axis. Proceeding in the same way, it can be shown that

$$I_Y = \bar{I}_y + Aa^2 \tag{5b}$$

Equations (5a) and (5b) represent the so-called *parallel-axis theorem* for moments of inertia of plane figures. In words, *the moment of inertia of a plane area with respect to any axis in its plane is equal to the moment of inertia with respect to a parallel centroidal axis plus the product of the total area and the square of the distance between the two axes.* We see that the further an axis is from the centroid of the area, the greater the moment of inertia of the area with respect to that axis.

Adding Eqs. (5a) and (5b) together and observing from Eq. (4) that $I_x + I_y = J_0$ while $\bar{I}_x + \bar{I}_y + \bar{I}_c$, , and from Fig. AI.12 that $a^2 + b^2 = d^2$, we obtain

$$\bar{J}_0 = \bar{J}_c + Ad^2 \tag{6}$$

Thus the parallel-axis theorem holds also for polar moments of inertia.

Using the parallel-axis theorem, various moments of inertia of a plane figure can readily be calculated without integration if the corresponding centroidal moment of inertia is already known. In Fig. AI.3, for example, we find for the moment of inertia of a rectangle with respect to its base

$$I_{x_1} = \frac{ba^3}{12} + ba\left(\frac{a}{2}\right)^2 = \frac{ba^3}{3}$$

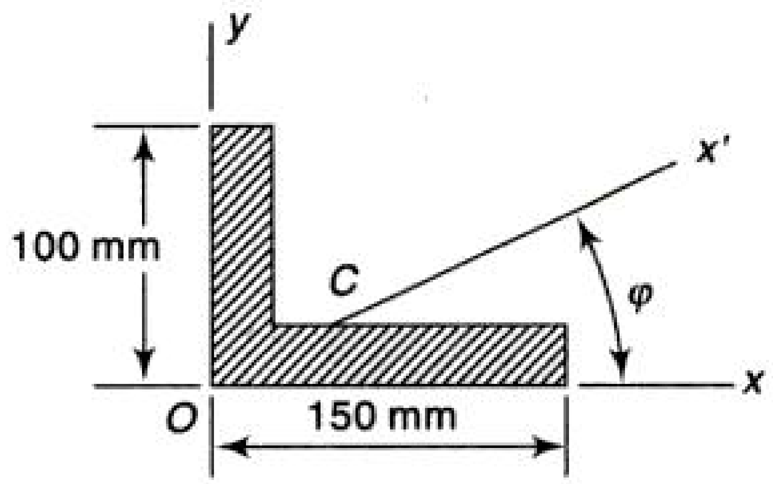

Fig. AI.A

2. Calculate the angle φ defining the direction of principal axes through the centroid C of the angle section shown in Fig. AI.F if each leg of the angle is 25 mm wide.
   (*Ans.* $\varphi = 67°26'$)
3. Calculate the principal moments of inertia of the angle section shown in Fig. AI.F with respect to the centroidal axes in the plane of the figure. Each leg is 25 mm wide.
   (*Ans.* $= 1.37 \times 10^7$ mm^4; $\bar{I}_{y'} = 2.6 \times 10^6$ mm^4)
4. Calculate the product of inertia $\bar{I}_{x'y'}$ of the rectangle shown in Fig. AI.B if $a = 250$ mm, $b = 150$ mm. (*Ans.* $\bar{I}_{x'y'} = -5.5 \times 10^4$ mm^4)

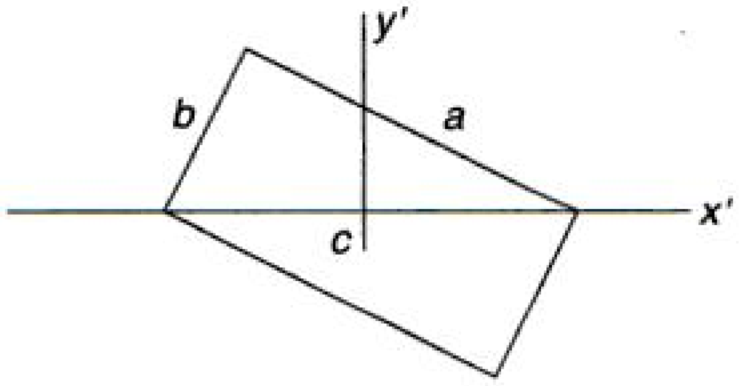

Fig. B

5. Calculate the angle φ defining the direction of principal axes through point O for the right triangle shown in Fig. AI.17, if $a = 400$ mm, $b = 200$ mm.
   (*Ans.* $\varphi = 60°07'$)

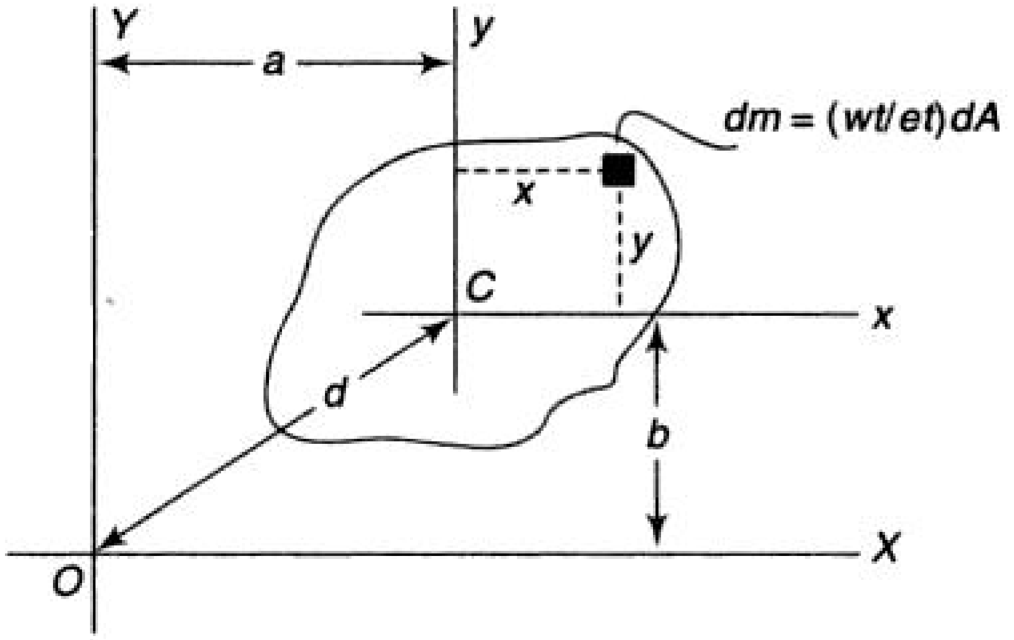

Fig. AII.21

Using the parallel-axis theorem together with the last of Eq. (c), we find, for example, that the polar moment of inertia of a thin circular plate of mass m and radius a with respect to an axis Z through a point O on its circumference is

$$I_Z = m\left(\frac{a^2}{2} + a^2\right) = \frac{3}{2}ma^2 \tag{e}$$

PROBLEM SET AII.7

1. A slender piece of wire of mass 70 g is bent in the shape shown in Fig. A. Calculate its moments of inertia I_x and I_y.
(*Ans.* $I_x = 58.25 \times 10^{-6}$ kg m^2; $I_y = 448.75 \times 10^{-6}$ kg m^2)

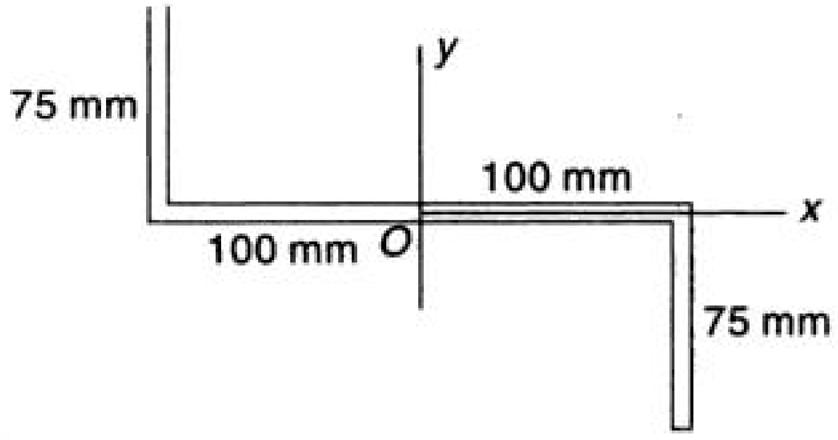

Fig. AII.A

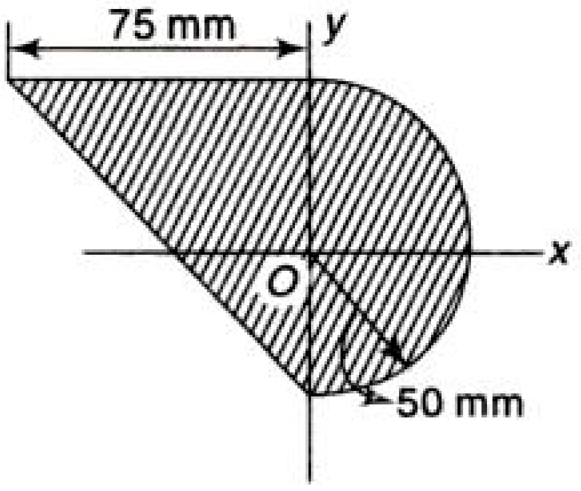

Fig. AII.B

2. Calculate the moment of inertia I_y of a homogeneous thin plate having the dimensions shown in Fig. AII.B. The total mass of the plate is m= 12.28 kg.
(*Ans.* $I_y = 99 \times 10^{-6}$ kg m^2)
3. Find the moment of inertia of a homogeneous triangular plate of mass m with respect to its base by using formula (c), for a plane triangular figure.
(*Ans.* $I_x = ma^2/6$)
4. Determine the moment of inertia of a homogeneous regular hexagonal lamina having mass m and sides of length a, with respect to a diagonal. (*Ans.* $I_d = (5/24)\,ma^2$)
5. Determine the moment of inertia I_x of a thin plate having the Z shape shown in Fig. 18 if the total mass of plate is $m = 25$ kg and $a = 125$ mm, $b = 75$ mm, $a_1 = 75$ mm, $b_1 = 50$ mm. (*Ans.* $I_x = 4.8 \times 10^{-4}$ kg m^2)

Referring again to Fig. AII.23, the product of inertia of the body with respect to the axes u and v can be calculated from the formula[2]

$$\begin{aligned} I_{uv} = -I_x \cos \alpha_1 \cos \alpha_2 - I_y \cos \beta_1 \cos \beta_2 - I_z \cos \gamma_1 \cos \gamma_2 \\ +I_{xy}(\cos \alpha_1 \cos \beta_2 + \cos \alpha_2 \cos \beta_1) \\ +I_{xz}(\cos \alpha_1 \cos \gamma_2 + \cos \alpha_2 \cos \gamma_1) \\ +I_{zy}(\cos \beta_1 \cos \gamma_2 + \cos \beta_2 \cos \gamma_1) \end{aligned} \tag{23}$$

Similar expression can be written for the products of inertia I_{vw} and I_{uw}.

If x, y, z are principal axes of the body, the products of inertia I_{xy}, I_{xz}, I_{yz} vanish and Eq. (23) reduces to the simpler form

$$I_{uv} = -I_x \cos \alpha_1 \cos \alpha_2 - I_y \cos \beta_1 \cos \beta_2 - I_z \cos \gamma_1 \cos \gamma_2 \tag{23a}$$

For the case of a lamina coinciding with the xy plane, we have for axes u, v, also in the plane of the lamina,

$$\cos \gamma_1 = \cos \gamma_2 = 0 \text{ and } I_{xz} = I_{yz} = 0$$

Under these conditions, Eq. (23) reduces to

$$\begin{aligned} I_{uv} = -I_x \cos \alpha_1 \cos \alpha_2 - I_y \cos \beta_1 \cos \beta_2 \\ + I_{xy} (\cos \alpha_1 \cos \beta_2 + \cos \alpha_2 \cos \beta_1) \end{aligned} \tag{23b}$$

Equation (23b) will be fousnd to agree with Eq. (12), if we note that between our present notations and those used in Fig. AI.19, there is the following correspondence:

$$\alpha_1 = \phi, \quad \alpha_2 = 90° = \phi, \quad \beta_1 = 90° - \phi, \quad \beta_2 = \phi$$

Equations (22) and (23) are very helpful in the calculation of moments of inertia and products of inertia of material bodies with respect to various axes for which direct integration would become very difficult. Consider, for example, the solid right circular cylinder of mass m, radius a, and length l, as shown in Fig. AII.25, and assume that we require the moment of inertia I_u and the product of inertia I_{uv} with respect to inclined axes u and v in an axial plane of symmetry as shown. These quantities, for instance, would be required in discussing rotation of the cylinder about the diagonal axis AB.

We begin with a calculation of the moment of inertia I_u. Taking principal axes x, y, z through the center of gravity C of the cylinder, and using Eqs (g) and (j), we have

$$I_x = m\frac{a^2}{2} \qquad I_y = I_z = m\left(\frac{a^2}{4} + \frac{l^2}{12}\right) \tag{q}$$

and since x, y, z are principal axes, $I_{xy} = I_{xz} = I_{yz} = 0$, and we may use Eq. (22a). Referring to Fig. AII.25, we see that

$$\cos \alpha_1 = \frac{1}{\sqrt{l^2 + 4a^2}}, \quad \cos \beta_1 = \frac{2a}{\sqrt{l^2 + 4a^2}}, \quad \cos \gamma_1 = 0 \tag{r}$$

---

[2] The derivation of this formula is omitted, although it can be made in a manner similar to that used in the derivation of Eq. (12), for the case of a plane figure.

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}}$$

will ensue. Suppose now that the rotor has an eccentric mass attached to its rim at A as shown. Such an eccentric rotating mass will produce a centrifugal force acting as shown in the figure. If Q_0 is the magnitude of this force, its vertical component, acting along the line of motion, represents the disturbing force in this case and we have[1]

$$Q = Q_0 \cos \omega t \tag{b}$$

where ω is the uniform angular speed of the rotor in radians per second and time t is measured from the instant when the centrifugal force Q_0 acts vertically downward.

We now replace Q in Eq. (a) by expression (b). Then dividing both sides of the equation by W/g and using the notations

$$\frac{k}{m} = \frac{kg}{W} = p^2, \quad \frac{Q_o}{m} = \frac{Q_0 g}{W} = q_0 \tag{c}$$

the latter of which is seen to represent the maximum value of the disturbing force per unit of vibrating mass, we obtain

$$\ddot{x} + p^2 x = q_0 \cos \omega t \tag{24}$$

This is the differential equation of *forced vibrations*.

Periodic Ground Motion

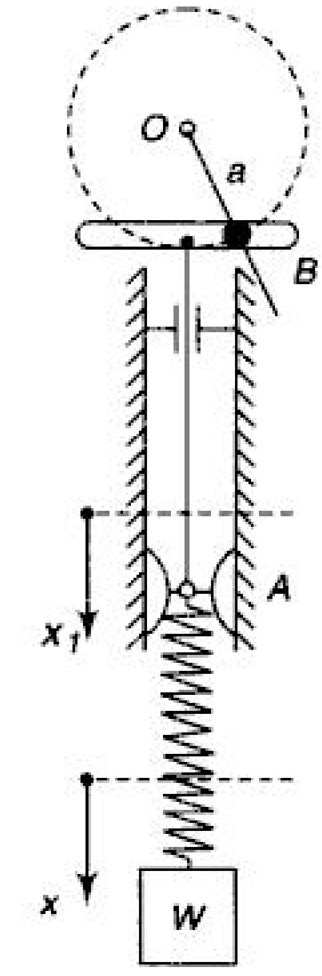

Fig. AIII.27

Before proceeding with the solution of Eq. (24), we shall mention another system (Fig. AIII.27), an analysis of which leads to the same differential equation.

Consider the weight W of mass m ($W = mg$) suspended from the lower end of a helical spring, the upper end of which is attached to a crosshead A which, by virtue of its connection with the rotating crank OB, performs the simple harmonic motion

$$x_1 = a \cos \omega t \tag{d}$$

where a is the length of the crank OB and w is the uniform angular speed with which it rotates. This forced motion of the crosshead A is called the *ground motion*. The time t is measured from the instant when the crank is vertically down, and hence the displacement x_1 of the crosshead is given with respect to its middle position. Owing to such motion of the upper end of the spring, the suspended weight W will also move, and we shall measure its displacement x from the position of static equilibrium when the crosshead A is in its middle position. In general x will be different from x_1, and it is evident that the difference $x - x_1$ between these two displacements represents the elongation of the spring over and above that existing for equilibrium conditions. Thus the equation of motion becomes

---

[1] It is assumed that the amplitude of viberation is small to that the path of point A may be considered to be a circle.

differ much from the static deflection produced by the maximum value of the disturbing force. This fact is of importance in the design of various types of *pressure indicators* used for measuring variable forces such as steam pressure in the cylinder of an engine. Such an instrument usually consists of a small piston attached to a spring, as shown in Fig. AIII.30.

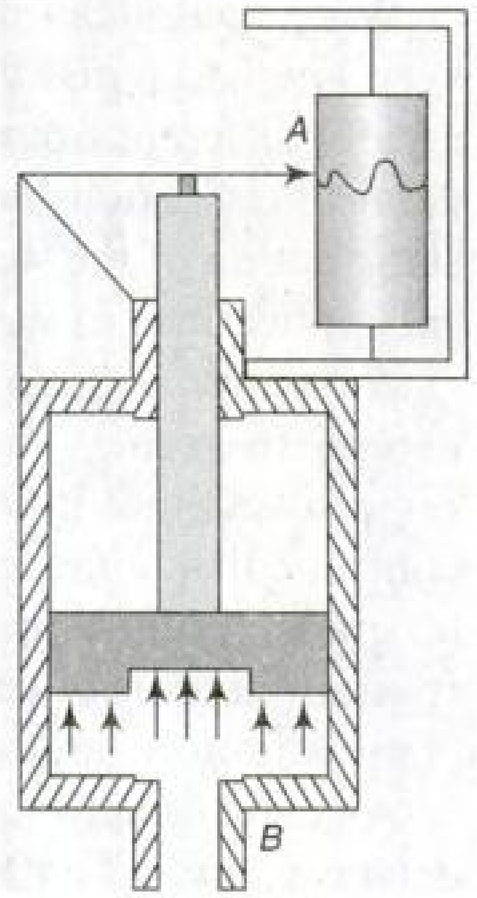

Fig. AIII.30

Thus we have a system, the natural frequency of which depends upon the spring constant k and the piston weight W. The instrument is connected to the cylinder of an engine by the tube B so that the piston is at all times subjected to the same pressure existing in the engine cylinder. Since this pressure is variable, it produces forced vibrations of the piston which are recorded on a rotating cylinder by the pencil A. In order that the displacement of the piston will always be approximately the same as would be produced by the same pressure acting statically, it is necessary that the natural frequency of the indicator system be many times greater than the frequency of fluctuations in the variable pressure that is to be measured. Under such conditions the ratio ω/p will be small and the value of the magnification factor will differ but little from unity. Thus the ordinates of the record can be taken as proportional to the pressure. It is seen that the general requirements for the instrument are a light-weight piston and a stiff spring, as this combination results in a high natural frequency.

Vibrograph

Let us consider now the condition of forced vibration well above resonance, where the ratio ω/p is large. This condition, as we have already seen, is characterized by very small amplitudes and a half-cycle phase difference between the disturbance and the forced vibration. These facts are utilized in the design of such instruments as the *vibrograph* and *seismograph* for measuring vibrations. An instrument of this kind is shown in Fig. AIII.31.

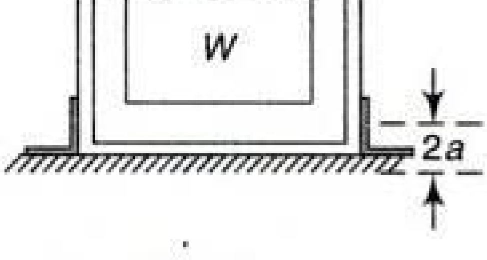

Fig. AIII.31

It consists essentially of a frame from which a heavy weight W is suspended by flexible springs. A recording dial A is arranged, as shown, to register any relative motion between the suspended weight and the frame. When the frame is fastened to any vertically vibrating body, say the bearing of a large turbine or generator, forced vibrations of the suspended weight W will be produced. If the natural frequency of the instrument is very low compared with the impressed frequency, represented in this case by the rps of the turbine or generator, the suspended weight W will practically stand still in space and hence the dial will show with good accuracy the absolute value of the vertical motion of the bearing to which the frame is bolted. The instrument can easily be adapted to the measurement of horizontal vibrations by using horizontal springs.

ing body its inertia force. Then, during rotation, the pressures on the bearings produced by these inertia forces can be calculated by using equations of statics. In writing these equations, we shall use a system of rectangular coordinate axes x, y, z, the origin of which coincides with the axis of rotation and shall assume that during rotation the x and y axes rotate together with the body. Taking at any point, distance r from the axis of rotation, a particle of mass dm and observing that the body is rotating uniformly, we conclude that the element has only radial acceleration $\omega^2 r$ and that the inertia force acting on it is $\omega^2 r\, dm$, directed as shown in the figure. The projections of this inertia force on the x and y axes are, respectively,

$$\omega^2 x\, dm \qquad \text{and} \qquad \omega^2 y\, dm \tag{a}$$

and its moment with respect to the same axes are, respectively,

$$-\omega^2 yz\, dm \quad \text{and} \quad \omega^2 xz\, dm \tag{b}$$

where the signs of moments are determined in accordance with the right-hand rule. The unknown reactions at the bearings can be resolved into components X_a, Y_a, and X_b, Y_b, as shown in the Fig. AIII.36.

For determining these components of the reactions, we write four equations of statics by equating to zero the sums of the projections of all forces on the x and y axes, and likewise the sums of moments of all forces with respect to the same axes. Thus, by using expressions (a) and (b), we obtain

$$\begin{aligned} &X_a + X_b + \omega^2 \textstyle\int x\, dm = 0 \\ &Y_a + Y_b + \omega^2 \textstyle\int y\, dm = 0 \\ &-Y_b\, l + \omega^2 \textstyle\int xz\, dm = 0 \\ &X_b\, l + \omega^2 \textstyle\int xz\, dm = 0 \end{aligned} \tag{c}$$

Introducing the notations[1]

$$\int x\, dm = \frac{W}{g} x_c = m x_c;\ \int y\, dm = \frac{W}{g} y_c = m y_c;$$

$$\int yz\, dm = I_{yz};$$

$$\int xz\, dm = I_{nz}$$

where W/g or m is the total mass of the body and x_c, y_c, the coordinates of its center of gravity, this system of equations can be put in the form

$$X_a + X_b = -\omega^2 \frac{W}{g} m x_c = -\omega^2 m x_c$$

$$Y_a + Y_b = -\omega^2 \frac{W}{g} m y_c = -\omega^2 m y_c$$

---

[1] The first two integrals are simply the *statical moments* of the body with respect to the yz and xz planes, respectively, and are well known from statics. The second two integrals are called *products of inertia* of the body with respect to the y, z and x, z axes, respectively. They are dimensionally similar to *moments of inertia* and are fully discussed in Appendix II.

fact some pulsating forces act on the bearings during rotation. To eliminate these forces completely, it is necessary to balance such rotors by adding proper correction masses/weights after the body has been cast and machined.

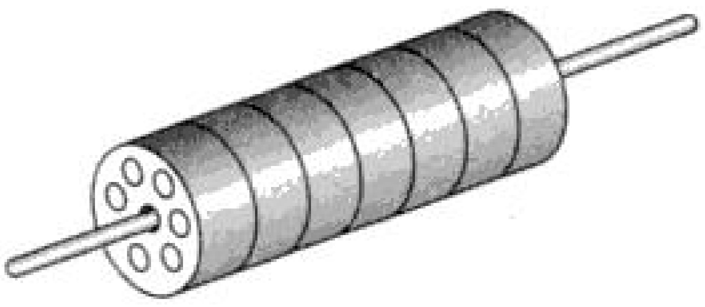

Fig. AIV.43

Consider, for example, the rotor shown in Fig. AIV.43. If we imagine this rotor to be divided into a number of thin disks, as shown, the center of gravity of each disk, owing to slight imperfections, will not be exactly on the axis of rotation and hence we obtain, during rotation, a system of radial inertia forces analogous to the case represented in Fig. AIV.41. It follows then that such unbalance can be completely equilibrated by two properly placed correction masses/weights, as discussed above. In the case of large rotors, as in electric machinery, the ends of the rotors usually have special holes along the circumference in which correction masses/weights can conveniently be placed. In such a case these end planes of the rotor would be chosen as the correction planes. In dealing with unbalance due to imperfections, however, we cannot calculate the proper correction masses weights since the unbalance is of an unknown nature, so it is necessary to accomplish balancing by a method of trial and error. This is usually done with the aid of special balancing machines.

Balancing Machines

Figure AIV.44 shows a horizontal bed supported by a fulcrum C and a spring S allowing small rotational oscillations about the axis through C normal to the plane of the figure.

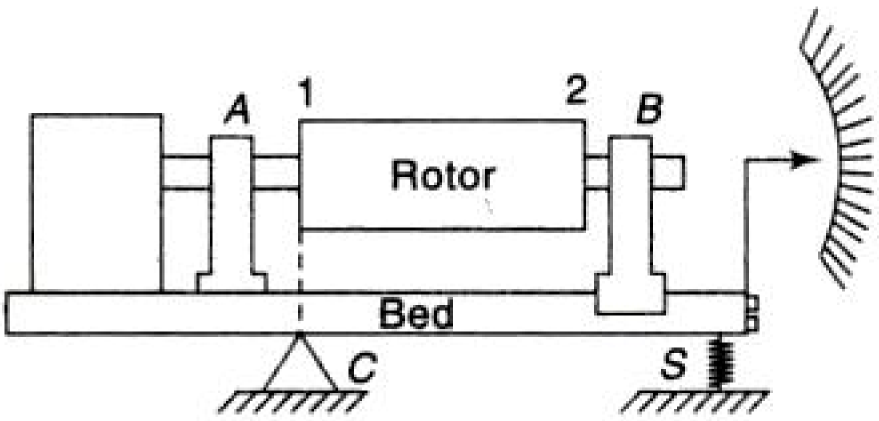

Fig. AIV.44

Such a system has a definite natural frequency of vibration, as discussed in Section 9.6, The bed carries two adjustable bearing pedestals A and B in which the rotor to be balanced can be mounted and connected to a variable-speed drive D as shown. The rotor is placed in these bearings with one correction plane directly over the fulcrum, and the drive speed is adjusted until the unbalance in the rotor comes into resonance with the natural frequency of the system. This condition will be recognized by the rather large amplitude of vibration of the bed. A correction mass m_2/weight W_2 is now placed in correction plane 2 and adjusted in both amount and angular position[3] until the rotor runs without producing any

---

[3] This can be done by trial and error, although most balancing machines have more or less elaborate devices to aid the operator in a more rapid selection of the proper device to aid the operator in a more rapid selection of the proper weight and its angular position.

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